7.6 Right off the bat, we notice that these are spin-1, and thus Bosons. Perhaps redundantly, we are told that the spatial part of their state vectors are symmetric. Nature does not allow states to be partially symmetric, and partially antisymmetric. Thus the spin states will have to be completely symmetric, which means interchanging particles will leave the state unchanged.

(a) I'll label the states $|i\rangle$, $|ii\rangle$, and $|iii\rangle$ per the problem numbering.

$$\boxed{|i\rangle = |+\rangle|+\rangle|+\rangle}$$ ($\langle i|i\rangle = 1$, so this is properly normalized)

$$J_-|i\rangle = J_-|33\rangle = J_{1-}|+\rangle|+\rangle|+\rangle + J_{2-}|+\rangle|+\rangle|+\rangle + J_{3-}|+\rangle|+\rangle|+\rangle$$

$$\hbar\sqrt{6}\,|ii\rangle = \hbar\sqrt{2}\,|0\rangle|+\rangle|+\rangle + \hbar\sqrt{2}\,|+\rangle|0\rangle|+\rangle + \hbar\sqrt{2}\,|+\rangle|+\rangle|0\rangle$$

$$\Rightarrow \boxed{|ii\rangle = \frac{1}{\sqrt{3}}\left(|0\rangle|+\rangle|+\rangle + |+\rangle|0\rangle|+\rangle + |+\rangle|+\rangle|0\rangle\right)}$$

etc.

$$\boxed{|iii\rangle = \frac{1}{\sqrt{6}}\left(|0\rangle|+\rangle|-\rangle + |0\rangle|-\rangle|+\rangle + |+\rangle|0\rangle|-\rangle + |+\rangle|-\rangle|0\rangle + |-\rangle|+\rangle|0\rangle + |-\rangle|0\rangle|+\rangle\right)}$$

Notice how all the states are added so as to make them symmetric

i) total spin = 3. ($|i\rangle = |3,3\rangle$)

ii) Since $[\vec{J}^2, \hat{J}_-] = 0$, $J^2 J_-|i\rangle = J_- J^2|i\rangle \Rightarrow 3J_-|i\rangle = J^2|ii\rangle$

$\Rightarrow$ total spin = 3 as well. ($|ii\rangle = |3,2\rangle$)

iii) For this one, we could use the lowering operator twice more, but then we'd have to subtract away some of the states. (3 of them, precisely)

$|iii\rangle = |3,0\rangle - |\text{something}\rangle$.

The states we get rid of are (unnormalized) $|-\rangle|+\rangle|+\rangle$, $|+\rangle|-\rangle|+\rangle$, $|+\rangle|+\rangle|-\rangle$ $|1,1\rangle$ states.

in other words, we subtracted away some

Combo of total spin 3 & 1

(b) i) impossible. There is no other way to arrange $|+\rangle|+\rangle|+\rangle$

ii) impossible. We still have repeated (identical) particles. (See text after 7.5.7).

iii) sadly, this one is possible.

$$\boxed{|iii\rangle = \frac{1}{\sqrt{6}}\left(|-\rangle|0\rangle|+\rangle - |0\rangle|-\rangle|+\rangle + |0\rangle|+\rangle|-\rangle - |+\rangle|0\rangle|-\rangle + |+\rangle|-\rangle|0\rangle - |-\rangle|+\rangle|0\rangle\right)}$$

total spin = $\emptyset$ per eq. 7.5.5

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