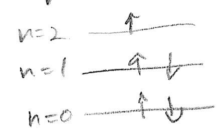
7.2 (a) Given N identical spin-$\frac{1}{2}$ particles in 1-D SHO. Find Ground State & Fermi Energies.

Each state has energy $E_n = (n+\frac{1}{2})\hbar\omega$

Figuring out how many energy levels get filled, think about filling boxes w/ 1 or 2 particles, starting from the bottom.



All we have to do to get the ground state energies is to add up the N lowest energy levels.

$$\therefore \quad E_{g.s.odd} = \sum_{n=0}^{n_{max,o}} (n+\tfrac{1}{2})\hbar\omega$$

$$= \sum_{n=0}^{n_{max,odd}} n\hbar\omega + \sum_{n=0}^{n_{max,o}} \tfrac{1}{2}\hbar\omega$$

The first term is

$$\underbrace{(0+0+1+1+2+2+3+3+\cdots+n_{max})}_{N\ \text{terms}}\hbar\omega$$

$$= \left(\frac{N-1}{2}\right)^2 \hbar\omega$$

and the second term is $\sum_{n=0}^{n_{max,o}} \frac{1}{2}\hbar\omega = \frac{N\hbar\omega}{2}$

$$\Rightarrow E_{g.s\ odd} = \hbar\omega\left[\left(\frac{N-1}{2}\right)^2 + \frac{N}{2}\right]$$

$$= \hbar\omega\left[\frac{N^2}{4} + \frac{1}{4} - \frac{2N}{4} + \frac{N}{2}\right]$$

$$\Rightarrow \boxed{E_{g.s.\ odd} = \left(\frac{N^2}{4} + \frac{1}{4}\right)\hbar\omega}$$

$$E_{g.s.even} = \sum_{n=0}^{n_{max,even}} (n+\tfrac{1}{2})\hbar\omega$$

The first term is

$$\underbrace{0+0+1+1+2+2+3+3+\cdots+n_{max}+n_{max}}_{N\ \text{terms}}$$

$$= \left(\frac{N}{2}\right)^2 - \frac{N}{2}$$

$$\Rightarrow E_{g.s.\ even} = \hbar\omega\Big[\underbrace{\left(\frac{N}{2}\right)^2 - \frac{N}{2}}_{\sum_{n=0}^{n_{max,e}} n} + \underbrace{\frac{1}{2}N}_{\sum_{n=0}^{n_{max,e}} \frac{1}{2}}\Big]$$

$$\Rightarrow \boxed{E_{g.s.\ even} = \frac{N^2\hbar\omega}{4}}$$

Fermi energies can be found by just taking the energy associated w/ $n_{max,odd}$ & $n_{max,even}$:

$$n_{max,odd} = \frac{N-1}{2}$$

$$\Rightarrow E_{f,odd} = \left[\left(\frac{N-1}{2}\right) + \frac{1}{2}\right]\hbar\omega$$

$$\boxed{E_{f,odd} = \frac{N}{2}\hbar\omega}$$

$$n_{max,\ even} = \frac{N}{2} - 1$$

$$E_{f,even} = \left[\frac{N}{2} - \frac{1}{2}\right]\hbar\omega$$

$$\boxed{E_{f,even} = \frac{N}{2}\hbar\omega - \frac{\hbar\omega}{2}}$$

(b) When N is very large, we neglect the terms in the ground state energy w/o N^2, and those in the Fermi Energy w/o N.

$$E_{g.s\,even} = E_{g.s\,odd} = \boxed{E_{g.s.} \simeq \frac{N^2}{4}\hbar\omega}$$

$$E_{fermi\ even} = E_{f,odd} = \boxed{E_f \simeq \frac{N}{2}\hbar\omega}$$

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