6.5) Weak Yukawa potential: $V = \frac{V_0 e^{-\mu r}}{\mu r}$, $\mu > 0$

(a) Given $f^{(1)}(\theta) = \frac{-2mV_0}{\hbar^2 \mu} \frac{1}{(2k^2(1-\cos\theta) + \mu^2)}$

we can expand $f(\theta)$ in terms of Legendre Polynomials (see Zettili 11.99)

$$f(\theta) = \frac{1}{k}\sum_{\ell=0}^{\infty}(2\ell+1)e^{i\delta_\ell}\sin\delta_\ell \, P_\ell(\cos\theta)$$

The Yukawa potential is weak, so $\delta_\ell \ll 1$

(*) $f(\theta) \approx \frac{1}{k}\sum_\ell (2\ell+1)\,\delta_\ell \, P_\ell(\cos\theta)$, where I used the small angle approx.

Then recall $\int_{-1}^{1} dx\, P_\ell(x) P_{\ell'}(x) = \frac{2}{2\ell+1}\delta_{\ell,\ell'}$ (†)

Now integrate (*) with $\int_{-1}^{1} P_{\ell'}(\cos\theta)\, d(\cos\theta)$

$$\int_{-1}^{1} f(\theta) P_{\ell'}(\cos\theta)\, d(\cos\theta) = \frac{1}{k}\int_{-1}^{1}\sum_\ell (2\ell+1)\,\delta_\ell\, P_\ell(\cos\theta) P_{\ell'}\cos(\theta')\, d\cos\theta')$$

Collapse the RHS using (†)

$$\int_{-1}^{1} f(\theta) P_{\ell'}(\cos\theta)\, d(\cos\theta) = \frac{1}{k}\sum_\ell \cancel{(2\ell+1)}\,\delta_\ell \frac{2}{\cancel{(2\ell+1)}}\delta_{\ell,\ell'} = \frac{2\delta_{\ell'}}{k}$$

Plug in $f^{(1)}(\theta)$ for $f(\theta)$, and let $x \equiv \cos\theta$

$$\int_{-1}^{1} \frac{-2mV_0}{\hbar^2\mu}\frac{1}{2k^2(1-x)+\mu^2} P_{\ell'}(x)\,dx = \frac{2\delta_{\ell'}}{k}$$

$$\frac{-2mV_0}{\hbar^2 k^2 \mu}\frac{1}{2}\int_{-1}^{1}\frac{P_{\ell'}(x)\,dx}{1+\frac{\mu^2}{2k^2}-x} = \frac{-2mV_0}{\hbar^2 k^2 \mu} Q_{\ell'}\left(1+\frac{\mu^2}{2k^2}\right) = \frac{2\delta_{\ell'}}{k}$$

Let $\ell' \to \ell$, and solve for δ_ℓ.

$$\boxed{\delta_\ell = \frac{-mV_0}{\hbar^2 k \mu} Q_\ell\left(1+\frac{\mu^2}{2k^2}\right)}$$

(b) i) Note that $1+\frac{\mu^2}{2k^2}$. Using the given expansion for Q_ℓ, clearly $Q_\ell\left(1+\frac{\mu^2}{k^2}\right) > 0$.

Also $\frac{-m}{\hbar^2 \mu k} < 0$ always. Thus δ_ℓ has the opposite sign of V_0.

ii) $\frac{1}{\mu} \sim$ range of the potential. Given $\lambda_{db} \gg \frac{1}{\mu} \Rightarrow \frac{1}{k} \gg \frac{1}{\mu} \Rightarrow 1 \ll \frac{\mu}{k}$

$\therefore 1 + \frac{\mu^2}{2k^2} \approx \frac{\mu^2}{2k^2} \Rightarrow \delta_\ell \approx \frac{-mV_0}{\hbar^2 k\mu} Q_\ell\left(\frac{\mu^2}{2k^2}\right)$

Using the given expansion, take it to first term.

$$Q_\ell\left(\frac{\mu^2}{2k^2}\right) \approx \frac{\ell!}{(2\ell+1)!!}\frac{1}{\left(\frac{\mu^2}{2k^2}\right)^{\ell+1}} = \frac{\ell!}{(2\ell+1)!!}\frac{1}{\mu^{2\ell+2}} 2^{\ell+1} k^{2\ell+2}$$

$$\therefore \boxed{\delta_\ell = \frac{-mV_0}{\hbar^2 \mu^{2\ell+3}} \frac{\ell!\, 2^{\ell+1}}{(2\ell+1)!!} k^{2\ell+1}}$$