6.4 Since all dependance is radial, we'll solve the radial Schrödinger Equation (6.4.55)

$$u_\ell''(r) + \left(k^2 - \frac{2m}{\hbar^2}V - \frac{\ell(\ell+1)}{r^2}\right)u_\ell = 0 \qquad \text{where } u_\ell = rA_\ell(r) \quad (A \text{ is radial equation}).$$

$$\therefore \quad u_\ell''(r) + \Big(\underbrace{\frac{2Em}{\hbar^2} - \frac{2m}{\hbar^2}V}_{= k'^2} - \frac{\ell(\ell+1)}{r^2}\Big)u_\ell = 0 \qquad \text{where I used } E = \frac{\hbar^2 k^2}{2m}$$

Solve this under condition $u_\ell|_{r=0} = 0$

$$u_\ell''(r) + \left(k'^2 - \frac{\ell(\ell+1)}{r^2}\right)u_\ell(r) = 0 \rightarrow u_\ell = r j_\ell(k'r) \rightarrow A_\ell = j_\ell(k'r)$$

(note that we've killed off the neuman function part of the solution as it violates the boundary condition $u_\ell|_{r=0}=0$.)

By 6.4.53,
$$\beta_\ell = \left(\frac{r}{A_\ell}\frac{dA_\ell}{dr}\right)\Big|_{r=R} = \frac{r}{j_\ell(k'r)} j_\ell'(k'r)\cdot k'\Big|_{r=R} = \frac{Rk' j_\ell'(k'R)}{j_\ell(k'R)}$$

when ℓ is small (it is, as we shall see) we can use the recurrance relation for $j_\ell'(k'R)$

$$j_\ell'(k'R) = \ell j_\ell \frac{1}{k'R} - j_{\ell+1}$$

$$\Rightarrow \beta_\ell = \ell - (k'R)\frac{j_{\ell+1}(k'R)}{j_\ell(k'R)} = \ell - (k'R)\left[\frac{\frac{(k'R)^{\ell+1}}{(2\ell+3)!!}}{\frac{(k'R)^\ell}{(2\ell+1)!!}}\right] \quad \text{by } 6.4.65$$

$$\Rightarrow \beta_\ell = \ell - \frac{(k'R)^2}{2\ell+3}$$

Next, by 6.4.54,

$$\tan\delta_\ell = \frac{kR j_\ell'(kR) - \beta_\ell j_\ell(kR)}{kR n_\ell'(kR) - \beta_\ell n_\ell(kR)} = \frac{kR j_\ell'(kR) - \left[\ell - \frac{(k'R)^2}{2\ell+3}\right]j_\ell(kR)}{kR n_\ell'(kR) - \left[\ell - \frac{(k'R)^2}{2\ell+3}\right]n_\ell(kR)}$$

use the recursion relations again for j_ℓ' and n_ℓ':

$$\tan\delta_\ell = \frac{kR\left(\cancel{\frac{\ell}{kR}j_\ell(kR)} - j_{\ell+1}(kR)\right) - \cancel{\ell j_\ell(kR)} + \frac{(k'R)^2}{2\ell+3}j_\ell(kR)}{kR\left(\cancel{\frac{\ell}{kR}n_\ell(kR)} - n_{\ell+1}(kR)\right) - \cancel{\ell n_\ell(kR)} + \frac{(k'R)^2}{2\ell+3}n_\ell(kR)}$$

$$= \frac{-(2\ell+3)kR j_{\ell+1}(kR) + (k'R)^2 j_\ell(kR)}{-(2\ell+3)kR n_{\ell+1}(kR) + (k'R)^2 n_\ell(kR)} = \frac{(k^2-k'^2)(kR)^{2\ell+3}}{(2\ell+1)!!(2\ell+3)!!}\left[\cancel{R^2k'^2\frac{1}{(2\ell+3)(2\ell+1)}}_{\to 0} - 1\right]$$

$(R^2k'^2 \ll 1)$

$$\Rightarrow \tan\delta_\ell \approx \frac{-(k^2-k'^2)(kR)^{2\ell+3}}{(2\ell+1)!!(2\ell+3)!!} \qquad \text{Now plug in } k'^2 = \frac{2m}{\hbar^2}(E-V_0),\ k^2 = \frac{2mE}{\hbar^2}$$

$$\tan\delta_\ell \approx \delta_\ell = \frac{-2mV_0}{\hbar^2}\frac{(kR)^{2\ell+3}}{(2\ell+1)!!(2\ell+3)!!} \quad \text{since the potential is small.}$$

To the lowest order use 6.4.40: $f(\theta) \approx \frac{1}{k}e^{i\delta_0}\sin\delta_0 \Rightarrow |f(\theta)|^2 = \frac{\delta_0^2}{k^2}$

$$\sigma_{tot,0} = 4\pi|f(\theta)|^2 = \frac{4\pi}{k^2}\cdot\frac{4m^2V_0^2R^6k^6}{\hbar^4\,9} \longrightarrow \boxed{\sigma_{tot} = \left(\frac{16\pi}{9}\right)\frac{m^2V_0^2R^6}{\hbar^4}k^4}$$

off by k^4. oops.

Now raise energy slightly. Let's go to $\ell=1$.

Again by 6.4.40, $f(\theta) = \frac{\delta_0}{k} + \frac{3\delta_1\cos\theta}{k} = \frac{\delta_0}{k}\left(1 + \frac{3\delta_1\cos\theta}{\delta_0}\right)$

Noting $\frac{\delta_1}{\delta_0} = \frac{(kR)^5}{(kR)^3}\cdot\frac{3}{45} \Rightarrow f(\theta) = \frac{\delta_0}{k}\left(1 + \frac{1}{5}\cos\theta\,(kR)^2\right) \Rightarrow \frac{d\sigma}{d\Omega} = |f(\theta)|^2 = \delta_0^2\left(1 + \frac{2}{5}\cos\theta(kR)^2 + \dots\right)$

Comparing w/ $\frac{d\sigma}{d\Omega} = A + B\cos\theta \longrightarrow \boxed{\frac{B}{A} = \frac{2}{5}(kR)^2}$