5.9 (a) we have a state: $|n, \ell=1, m=0,\pm1\rangle$, and a perturbation $V=\lambda(x^2-y^2)$

Now we are looking for the zeroth-order energy eigenstates.

We currently have a 3-fold degeneracy ($|n,1,0\rangle$, $|n,1,-1\rangle$, and $|n,1,+1\rangle$ all have the same energy eigenvalue).

→ To break the degeneracy, we must find $\Delta_n^{(1)}$ – The first order energy shifts.

$$V_{n\ell m',n\ell m}=\Delta_n^{(1)}=\lambda{}^{(0)}\langle n,\ell,m|(x^2-y^2)|n,\ell m\rangle^{(0)}$$

$$=\lambda\begin{bmatrix}\langle n10|x^2-y^2|n10\rangle & \langle n10|x^2-y^2|n11\rangle & \langle n10|x^2-y^2|n1-1\rangle\\ \langle n11|x^2-y^2|n10\rangle & \langle n11|x^2-y^2|n11\rangle & \langle n11|x^2-y^2|n1-1\rangle\\ \langle n1-1|x^2-y^2|n10\rangle & \langle n1-1|x^2-y^2|n11\rangle & \langle n1-1|x^2-y^2|n1-1\rangle\end{bmatrix}$$

All but 3 of these entries are identically zero. Let's show this:

note: $x^2=r^2\sin^2\theta\cos^2\phi$, $y^2=r^2\sin^2\theta\sin^2\phi$ $\Rightarrow$ $x^2-y^2=r^2\sin^2\theta(\cos^2\phi-\sin^2\phi)$

also: $\langle r,\theta,\phi|n\ell m\rangle=R_{n\ell}(r)\,Y_{\ell m}(\theta,\phi)$

$$\Rightarrow \langle n\ell'm'|x^2-y^2|n\ell m\rangle=\int_0^\infty r^2dr\,R^*_{n\ell'}R_{n\ell}r^2\int d\Omega\,\sin^2\theta(\cos^2\phi-\sin^2\phi)\,Y^*_{\ell'm'}Y_{\ell m}$$

Thus:

- $\langle n10|x^2-y^2|n10\rangle=\int_0^\infty r^4dr|R_{n1}|^2\int d\Omega\,\frac{3}{4\pi}\cos^2\theta\sin^2\theta(\cos^2\phi-\sin^2\phi)=0$

  b/c This contains the integral $\int_0^{2\pi}d\theta\cos^2\theta\sin^3\theta=-\frac{1}{8}\cos\theta-\frac{1}{48}\cos(3\theta)+\frac{1}{80}\cos(5\theta)\Big|_0^{2\pi}=0$

- $\langle n11|x^2-y^2|n10\rangle=\int_0^\infty r^4dr|R_{n1}|^2\int d\Omega\sqrt{\frac{3}{8\pi}}\sqrt{\frac{3}{4\pi}}\sin\theta\cos\theta e^{i\phi}\sin^2\theta(\cos^2\phi-\sin^2\phi)=0$

  b/c this contains the integral $\int_0^{2\pi}d\theta\cos\theta\sin^4\theta=\frac{\sin^5(\theta)}{5}\Big|_0^{2\pi}=0$

- $\langle n1-1|x^2-y^2|n10\rangle=0$ as well because this only changes the former line by a minus sign.

- Thus $\langle n10|x^2-y^2|n11\rangle=\langle n10|x^2-y^2|n1-1\rangle=0$ b/c The θ integrals are all real.

- $\langle n11|x^2-y^2|n11\rangle=\int_0^\infty r^4dr|R_{n1}|^2\int d\Omega\,\frac{3}{8\pi}\sin^4\theta(\cos^2\phi-\sin^2\phi)=0$

  b/c $\int_0^{2\pi}d\theta\sin^5\theta=\frac{\sin^4\theta\cos\theta}{5}\Big|_0^{2\pi}=0$

- note that $\langle n1-1|x^2-y^2|n1-1\rangle=\langle n11|x^2-y^2|n11\rangle=0$.

Lastly:

$$\lambda\langle n1-1|x^2-y^2|n11\rangle=\lambda\langle n11|x^2-y^2|n1-1\rangle=\lambda\int_0^\infty r^4dr|R_{n1}|^2\int d\Omega\,\frac{3}{8\pi}\sin^4\theta(\cos^2\phi-\sin^2\phi)\equiv B$$

$$\Rightarrow V=\begin{bmatrix}0&0&0\\0&0&B\\0&B&0\end{bmatrix}$$

To get the zeroth order eigenstates, we diagonalize & find eigenvectors.

$\begin{bmatrix}0&B\\B&0\end{bmatrix}\longrightarrow\lambda=\pm B$ (eigenvalues)

so $V-\lambda I=\begin{bmatrix}-B&B\\B&-B\end{bmatrix}$ gives us eigenvectors $\frac{1}{\sqrt2}\begin{bmatrix}1\\1\end{bmatrix},\ \frac{1}{\sqrt2}\begin{bmatrix}1\\-1\end{bmatrix}$ $\Rightarrow$ Thus, The Zeroth-order Eigenstates are $\boxed{\frac{1}{\sqrt2}(|n11\rangle\pm|n,1,-1\rangle)}$ ↓

(b) In part a, we found the energy eigenstates to be

$$|+\rangle = \frac{1}{\sqrt{2}}(|n\,1\,1\rangle + |n\,1\,-1\rangle)$$
$$|-\rangle = \frac{1}{\sqrt{2}}(|n\,1\,1\rangle - |n\,1\,-1\rangle)$$

Prove $|+\rangle$ and $|-\rangle$ are eigenstates of time-reversal Θ

$$\Theta|+\rangle = \frac{1}{\sqrt{2}}\left[\Theta|n\,1\,1\rangle + \Theta|n\,1\,-1\rangle\right]$$

note $\Theta|l,m\rangle = (-1)^m|l,-m\rangle$ by 4.4.58

Thus

$$\boxed{\begin{aligned}\Theta|+\rangle &= \frac{1}{\sqrt{2}}\left[-|n\,1\,-1\rangle - |n\,1\,1\rangle\right] = -|+\rangle \checkmark \\ \Theta|-\rangle &= \frac{1}{\sqrt{2}}\left[-|n\,1\,-1\rangle + |n\,1\,1\rangle\right] = |-\rangle \checkmark\end{aligned}}$$

$\frac{10}{10}$

Checked