5.40 from 5.8.32 we have $\frac{d\sigma}{d\Omega} = \frac{4\pi^2\alpha\hbar}{m_e^2\omega} |\langle \vec{k}_f | e^{i(\omega/c)(\hat{n}\cdot\vec{x})} \hat{\varepsilon}\cdot\vec{p}|i\rangle|^2 \frac{m_e k_f L^3}{\hbar^2 (2\pi)^3}$

Then note that the matrix element is given by 5.8.33 in the ground-state H-atom case. replace the quantity in brackets w/ the wavefunction of a 3-d isotropic harmonic oscillator (ground state).

which is $\psi_0 = \left(\frac{m\omega_0}{\pi\hbar}\right)^{3/4} e^{-\frac{1}{2}x^2 m\omega_0/\hbar}$ (i.e. the prefactor is cubed as compared to the 1-D case).

$$\Rightarrow \langle k_f | e^{i(\omega/c)(\hat{n}\cdot\vec{x})} \hat{\varepsilon}\cdot\vec{p}|i\rangle = \hat{\varepsilon}\cdot\int d^3x \frac{e^{-i\vec{k}_f\cdot\vec{x}}}{L^{3/2}} e^{i(\frac{\omega}{c})(\hat{n}\cdot\vec{x})}(-i\hbar\vec{\nabla})\left[\left(\frac{m\omega_0}{\pi\hbar}\right)^{3/4} e^{-\frac{1}{2}x^2 m\omega_0/\hbar}\right]$$

$$= \frac{-i\hbar}{L^{3/2}}\left(\frac{m\omega_0}{\pi\hbar}\right)^{3/4} \hat{\varepsilon}\cdot\int d^3x\, \vec{\nabla} e^{-i\vec{k}_f\cdot\vec{x}} e^{-i(\frac{\omega}{c})(\hat{n}\cdot\vec{x})} e^{-\frac{1}{2}x^2 m\omega_0/\hbar}$$

Per the text following 5.8.33 (the result of integrating by parts). So The derivative brings down $-i k_f$

$$= \frac{-\hbar^2}{L^{3/2}}\left(\frac{m\omega_0}{\pi\hbar}\right)^{3/4} \hat{\varepsilon}\cdot\vec{k}_f \int d^3x\, e^{-i\vec{q}\cdot\vec{x}} e^{-\frac{1}{2}x^2 m\omega_0/\hbar}$$ where $q \equiv \vec{k}_f - \left(\frac{\omega}{c}\right)\hat{n}$ per 5.8.35

Thus from the top line $|\langle \vec{k}_f|\cdots|i\rangle|^2 = \left(\frac{m\omega_0}{\pi\hbar}\right)^{3/2} \frac{\hbar^4}{L^3}(\hat{\varepsilon}\cdot\vec{k}_f)^2 \left|\int d\vec{r}\, e^{-i\vec{q}\cdot\vec{r}} e^{-\frac{1}{2}r^2 m\omega_0/\hbar}\right|^2$

$$\Rightarrow \frac{d\sigma}{d\Omega} = \underbrace{(\hat{\varepsilon}\cdot\vec{k}_f)^2\left(\frac{m\omega_0}{\hbar\pi}\right)^{3/2}\frac{k_f\alpha\hbar^4}{2m_e\pi\omega}}_{\equiv\, \gamma} \left|\int_{-\infty}^{\infty} dx\, e^{-q_x x} e^{-\frac{1}{2}\frac{x^2}{\hbar} m\omega_0} \underbrace{\int_{-\infty}^{\infty} dy\, e^{-q_y y} e^{-\frac{1}{2}\frac{y^2}{\hbar} m\omega_0}}_{\to\, e^{-\frac{q_y^2\hbar}{2m\omega_0}}\sqrt{\frac{2\pi\hbar}{m\omega_0}}} \int_{-\infty}^{\infty} dz\, e^{-q_z z} e^{-\frac{1}{2}\frac{z^2}{\hbar} m\omega_0}\right|^2$$

These three integrals are evaluated using mathematica: e.g.

$$\Rightarrow \frac{d\sigma}{d\Omega} = \gamma\left(\sqrt{\frac{2\pi\hbar}{m\omega_0}}\right)^6 e^{\frac{-q^2\hbar}{m\omega_0}} = \gamma\left(\frac{2\pi\hbar}{m\omega_0}\right)^3 e^{\frac{-q^2\hbar}{m\omega_0}}$$

Now, $q^2 = k_f^2 - 2\left(\frac{\omega}{c}\right)\vec{k}_f\cdot\hat{n} + \frac{\omega^2}{c^2} = k_f^2 + \frac{\omega^2}{c^2} - 2\left(\frac{\omega}{c}\right)\cos\theta\, k_f$ (per figure Fig. 5.12)

$$= \gamma\left(\frac{2\pi\hbar}{m\omega_0}\right)^3 \exp\left[\frac{-\hbar}{m\omega_0}\left(k_f^2 + \frac{\omega^2}{c^2} - 2\frac{\omega k_f}{c}\cos\theta\right)\right]$$

$$= \gamma\left(\frac{2\pi\hbar}{m\omega_0}\right)^3 \exp\left[\frac{-\hbar}{m\omega_0}\left[k_f^2 + \left(\frac{\omega}{c}\right)^2\right]\right] \exp\left[\frac{2\hbar k_f\omega}{m\omega_0 c}\cos\theta\right]$$

now, $(\hat{\varepsilon}\cdot\vec{k}_f)^2 = k_f^2\sin^2\theta\cos^2\phi$ by 5.8.37

$$\Rightarrow \boxed{\frac{d\sigma}{d\Omega} = \frac{4\alpha\hbar^{8} k_f^3}{m^2\omega\omega_0}\sqrt{\frac{\pi\hbar}{m\omega_0}} \exp\left(-\frac{\hbar}{m\omega_0}\left[k_f^2 + \left(\frac{\omega}{c}\right)^2\right]\right) \sin^2\theta\cos^2\phi \exp\left[\left(\frac{2\hbar k_f\omega}{m\omega_0 c}\right)\cos\theta\right]}$$

(not sure what's up w/ the extra $\hbar^3$).

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