5.4

(a) given: $H_0 = \frac{P_x^2}{2m} + \frac{m\omega^2 x^2}{2} + \frac{P_y^2}{2m} + \frac{m\omega^2 y^2}{2} \equiv H_{0x} + H_{0y}$

we can analyze each dimension independantly.

$H_{0x} \Rightarrow E_{n_x} = (n_x + \frac{1}{2})\hbar\omega$, $\quad H_{0y} \Rightarrow E_{n_y} = (n_y + \frac{1}{2})\hbar\omega$

and $\quad E_{n_x n_y} = E_{n_x} + E_{n_y} = (n_x + \frac{1}{2})\hbar\omega + (n_y + \frac{1}{2})\hbar\omega$

The energy eigenfuctions are $|n_x\, n_y\rangle$, and the 3 lowest-lying states are $|00\rangle$, $|01\rangle$, and $|10\rangle$

$\Rightarrow \boxed{E_{00}^{(0)} = \hbar\omega, \quad E_{10}^{(0)} = E_{01}^{(0)} = 2\hbar\omega}$ 2-fold degeneracy in first excited state.

(b) Add a Perturbation: $V = \delta m\omega^2 xy$

Find the zeroth-order energy eigenstates, as well as $E_{00}^{(1)}$, $E_{10}^{(1)}$, and $E_{01}^{(1)}$, where $E_{n_x n_y}^{(1)} = E_{n_x n_y}^{(0)} + \Delta_{n_x n_y}^{(1)}$

Ground State

$\rightarrow$ Zeroth-order Energy Eigenkets (unpicturbed)

in general $\psi(x,y) = f(x)g(y)$ because we have assumed a seperable form.

$\psi_0(x) = \langle x|0\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-m\omega x^2/2\hbar}$ and $\psi_1(x) = \langle x|1\rangle = \left[\frac{4}{\pi}\left(\frac{m\omega}{\hbar}\right)^3\right]^{1/4} x e^{-m\omega x^2/2\hbar}$

by 2.3.30-31.

$$\boxed{\psi_0^{(0)}(x,y) = \langle x|0\rangle\langle y|0\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} e^{-m\omega(x^2+y^2)/2\hbar}}$$

$\rightarrow$ First-order energy shift: $\Delta_0^{(1)} = {}^{(0)}\langle 00|\delta m\omega^2 xy|00\rangle^{(0)} = \delta m\omega^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(\left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\right)^2 \left(e^{-m\omega(x^2+y^2)/2\hbar}\right)^2 xy\,dx\,dy$

$\Rightarrow \Delta_0^{(1)} = 0$ b/c $\int_{-\infty}^{\infty} xe^{-x^2}dx = 0$, since xe^{-x^2} is odd.

First Excited State

$\rightarrow$ First-order energy shift: $\Delta_1^{(1)} = \langle 1^{(0)}|V|1^{(0)}\rangle = \delta m\omega^2 \langle 1^{(0)}|xy|1^{(0)}\rangle$

This becomes a 2x2 matrix b/c of the 2-fold degeneracy.

$$\Delta_1^{(1)} = \begin{bmatrix} \langle 01|xy|01\rangle & \langle 01|xy|10\rangle \\ \langle 10|xy|01\rangle & \langle 10|xy|10\rangle \end{bmatrix} \delta m\omega^2$$

calculate the matrix elements:

- $\delta m\omega^2 \langle 01|xy|01\rangle = \delta m\omega^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \psi_1^*(y)\psi_0^*(x)\cdot\psi_1(y)\psi_0(x)\, xy\,dx\,dy$

$= \delta m\omega^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{2}{\pi}\left(\frac{m\omega}{\hbar}\right)^2 y^2 e^{-m\omega(x^2+y^2)/\hbar} xy\,dx\,dy = 0$ b/c the x-integral vanishes again.

By symmetry, $\langle 10|xy|10\rangle = \langle 01|xy|01\rangle = 0$.

- $\delta m\omega^2 \langle 10|xy|01\rangle = \delta m\omega^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \psi_1^*(x)\psi_0^*(y)\,\psi_1(y)\psi_0(x)\, xy\,dx\,dy$

$= \delta m\omega^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{2}{\pi}\left(\frac{m\omega}{\hbar}\right)^2 xy\, e^{-m\omega(x^2+y^2)/\hbar} xy\,dx\,dy = \delta m\omega^2 \frac{2}{\pi}\left(\frac{m\omega}{\hbar}\right)^2 \frac{\pi}{4}\left(\frac{\hbar}{m\omega}\right)^3 = \frac{\delta\omega\hbar}{2}$

Since these are real wavefuctions, $\langle 10|xy|01\rangle = \langle 01|xy|10\rangle = \frac{\delta\omega\hbar}{2}$

$$\Rightarrow \Delta_1^{(1)} = \frac{\delta\omega\hbar}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

5.4 cont.

Now I diagonalize $\Delta_1^{(1)} = \frac{\delta\omega\hbar}{2}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$. $|\Delta_1^{(1)} - \lambda I| = 0 \Rightarrow \begin{vmatrix}-\lambda & \frac{\delta\omega\hbar}{2}\\ \frac{\delta\omega\hbar}{2} & -\lambda\end{vmatrix} = 0$

$\Rightarrow \lambda^2 = \left(\frac{\delta\omega\hbar}{2}\right)^2 \Rightarrow \lambda_\pm = \pm\frac{\delta\omega\hbar}{2}$

Thus, the two energy shifts are $\Delta_{1\pm}^{(1)} = \pm\frac{\delta\omega\hbar}{2}$.

→ get the eigenkets.

$\Delta_1^{(1)} - \lambda_+ I = \frac{\delta\omega\hbar}{2}\begin{bmatrix}-1 & 1\\ 1 & -1\end{bmatrix} \Rightarrow \psi_{1+}^{(0)} = \frac{1}{\sqrt2}\begin{bmatrix}1\\1\end{bmatrix} = \frac{1}{\sqrt2}(|10\rangle + |01\rangle)$

$\Delta_1^{(1)} - \lambda_- I = \frac{\delta\omega\hbar}{2}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix} \Rightarrow \psi_{1-}^{(0)} = \frac{1}{\sqrt2}\begin{bmatrix}1\\-1\end{bmatrix} = \frac{1}{\sqrt2}(|10\rangle - |01\rangle)$

To sum it all up:

	Zeroth order energy eigenstate	1^{st} order energy
ground state:	$\psi_0^{(0)} = \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} e^{-m\omega(x^2+y^2)/2\hbar}$	$E_0^{(1)} = \hbar\omega$
1^{st} excited state:	$\psi_{1+}^{(0)} = \frac{1}{\sqrt\pi}\left(\frac{m\omega}{\hbar}\right) e^{-m\omega(x^2+y^2)/2\hbar}(x+y)$	$E_{1+}^{(1)} = 2\hbar\omega + \frac{\delta\omega\hbar}{2}$
	$\psi_{1-}^{(0)} = \frac{1}{\sqrt\pi}\left(\frac{m\omega}{\hbar}\right) e^{-m\omega(x^2+y^2)/2\hbar}(x-y)$	$E_{1-}^{(1)} = 2\hbar\omega - \frac{\delta\omega\hbar}{2}$

(c) $H_0 + V = \frac{P_x^2}{2m} + \frac{P_y^2}{2m} + m\omega^2\left(\delta xy + \frac{x^2}{2} + \frac{y^2}{2}\right) = \frac{P_x^2}{2m} + \frac{P_y^2}{2m} + \frac{m\omega^2}{2}\left(2xy\delta + x^2 + y^2\right)$

Now note we can write: $2xy\delta = \delta[(x+y)^2 - (x-y)^2]$

and $x^2+y^2 = (x+y)^2 + (x-y)^2$

$\Rightarrow H_0 + V = \frac{P_x^2}{2m} + \frac{P_y^2}{2m} + \frac{m\omega^2}{2}\left([(x+y)^2 - (x-y)^2]\delta + (x+y)^2 + (x-y)^2\right)$

$= \frac{P_x^2}{2m} + \frac{P_y^2}{2m} + \frac{m\omega^2}{2}\left[(1+\delta)(x+y)^2 + (1-\delta)(x-y)^2\right]$

Let $x' = x+y$, $y' = x-y$

$\Rightarrow H_0 + V = \frac{P_{x'}^2}{2m} + \frac{P_{y'}^2}{2m} + \frac{m\omega^2(1+\delta)x'^2}{2} + \frac{m\omega^2(1-\delta)y'^2}{2}$

This is just the Hamiltonian of an anisotropic 2D harmonic oscillator,

for the x' direction, $\omega \to \omega\sqrt{1+\delta}$

and for the y' direction, $\omega \to \omega\sqrt{1-\delta}$

20/20

$\Rightarrow E_{n_{x'}} = (n_{x'} + \tfrac12)\hbar\omega\sqrt{1+\delta}$, $E_{n_{y'}} = (n_{y'} + \tfrac12)\hbar\omega\sqrt{1-\delta}$

$\Rightarrow E_{n_{x'}n_{y'}} = (n_{x'} + \tfrac12)\hbar\omega\sqrt{1+\delta} + (n_{y'} + \tfrac12)\hbar\omega\sqrt{1-\delta}$

Thus

$E_{00} = \tfrac12\hbar\omega(\sqrt{1+\delta} + \sqrt{1-\delta}) \overset{\text{Binomial Approx}}{\simeq} \tfrac12\hbar\omega(1 + \tfrac12\delta + 1 - \tfrac12\delta) = \hbar\omega = E_0^{(1)}$ ✓

$E_{10} = \tfrac32\hbar\omega\sqrt{1+\delta} + \tfrac12\hbar\omega\sqrt{1-\delta} \simeq \tfrac32\hbar\omega + \tfrac34\hbar\omega\delta + \tfrac12\hbar\omega - \tfrac14\hbar\omega\delta = 2\hbar\omega + \frac{\hbar\omega\delta}{2} = E_{1+}^{(1)}$ ✓

$E_{01} = \tfrac32\hbar\omega\sqrt{1-\delta} + \tfrac12\hbar\omega\sqrt{1+\delta} \simeq 2\hbar\omega - \frac{\hbar\omega\delta}{2} = E_{1-}^{(1)}$ ✓

$\boxed{\text{They all check out!}}$