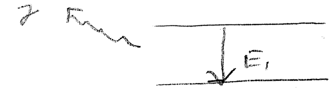

$\boxed{5.34}$ As in Section 5.8, we begin with the hamiltonian given to first order in 5.8.1: $H = H_0 + V$

$$H_0 = \frac{\vec{p}^2}{2m} + e\phi(\vec{x}), \quad V = \frac{-e}{mc}\vec{A}\cdot\vec{p}.$$

we express $\vec{A}$ using 5.8.5: $\vec{A} = A_0\hat{\varepsilon}\left[e^{i(\frac{\omega}{c})\hat{n}\cdot\vec{x} - i\omega t} + e^{-i(\frac{\omega}{c})\hat{n}\cdot\vec{x} + i\omega t}\right]$

$$\Rightarrow V = -\frac{e}{mc}A_0\hat{\varepsilon}\cdot\vec{p}\left[e^{i(\frac{\omega}{c})\hat{n}\cdot\vec{x} - i\omega t} + e^{-i(\frac{\omega}{c})\hat{n}\cdot\vec{x} + i\omega t}\right]$$

This takes the form of the Harmonic perturbation: $V = \mathcal{V}e^{i\omega t} + \mathcal{V}^\dagger e^{-i\omega t}$

In the emission case, we are lowering the state, so we use the lowering operator part,

$$\mathcal{V} = \frac{-e}{mc}A_0\hat{\varepsilon}\cdot\vec{p}\,e^{-i(\frac{\omega}{c})\hat{n}\cdot\vec{x} + i\omega t} = V_{ni}$$

Thus, $w_{i\to n} = \frac{2\pi}{\hbar}|V_{ni}|^2\delta(E_n - E_i - \hbar\omega) = \frac{2\pi}{\hbar}\frac{e^2|A_0|^2}{m^2c^2}|\langle n|e^{-i(\frac{\omega}{c})\hat{n}\cdot\vec{x}}\hat{\varepsilon}\cdot\vec{p}|i\rangle|^2\delta(E_n - E_i - \hbar\omega)$

In the dipole approx. (5.8.15), we have $e^{-i(\frac{\omega}{c})\hat{n}\cdot\vec{x}} \approx 1$

$$\therefore w_{i\to n} = \frac{2\pi}{\hbar}\frac{e^2|A_0|^2}{m^2c^2}|\hat{\varepsilon}\cdot\langle n|\vec{p}|i\rangle|^2\delta(E_n - E_i - \hbar\omega)$$

With $w_{i\to n}$ in terms of momentum, we're stuck. In order to write this in terms of angles, we'll want this in a position representation, and then convert to spherical harmonics. Following N. Zettili's Quantum Mechanics, 10.5,

$$[\hat{x}, \hat{H}_0] = [\hat{x}, \frac{p^2}{2m} + e\phi(\vec{x})] = [\hat{x}, p^2/2m] = \frac{1}{2m}\left([\hat{x}, p_x^2] + \cancel{[\hat{x}, p_y^2]} + \cancel{[\hat{x}, p_z^2]}\right)$$

$$= \frac{1}{2m}\left([\hat{x}, \hat{p}_x]\hat{p}_x + \hat{p}_x[\hat{x}, \hat{p}_x]\right) = \frac{2i\hbar\hat{p}_x}{2m} = \frac{i\hbar\hat{p}}{m}$$

$$\therefore \hat{\varepsilon}\cdot\langle n|\vec{p}|i\rangle = \frac{m}{i\hbar}\hat{\varepsilon}\cdot\left(\langle n|\vec{x}H_0|i\rangle - \langle n|H_0\vec{x}|i\rangle\right) = \frac{m}{i\hbar}(E_i - E_n)\hat{\varepsilon}\cdot\langle n|\vec{x}|i\rangle$$

B/c $H_0|k\rangle = E_k|k\rangle$

Thus, $w_{i\to n} = \frac{2\pi}{\hbar}\frac{e^2}{\cancel{m^2}c^2}|A_0|^2\frac{\cancel{m^2}}{\hbar^2}(E_i - E_n)^2|\hat{\varepsilon}\cdot\langle n|\vec{x}|i\rangle|^2\delta(E_n - E_i - \hbar\omega)$

In other words, $w_{i\to n} \propto |\hat{\varepsilon}\cdot\langle n|\vec{x}|i\rangle|^2$

Now, write $\vec{x}$ in terms of 3 unit vectors $\hat{a}$, $\hat{b}$, and $\hat{c}$, which are orthogonal:

$$\vec{x} = \frac{(a+ib)}{\sqrt{2}}\underbrace{\frac{(\hat{a} - i\hat{b})}{\sqrt{2}}}_{\equiv\,\hat{n}_a} + \frac{(a-ib)}{\sqrt{2}}\underbrace{\frac{(\hat{a} + i\hat{b})}{\sqrt{2}}}_{\equiv\,\hat{n}_b} + \underbrace{c\hat{c}}_{\equiv\,\hat{n}_c}$$

and recognize the coefficients as spherical Harmonics!

$$\vec{x} = |x|\sqrt{\frac{4\pi}{3}}\Big(\underbrace{Y_1^1\hat{n}_a + Y_1^{-1}\hat{n}_b}_{\sim\, Y_1^0} + Y_1^0\hat{n}_c\Big)$$

$\Rightarrow \vec{x} \propto Y_1^0 \sim \cos\theta.$

Thus, $\boxed{w_{i\to n} \propto \cos^2\theta}$

polarization direction?

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