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By 5.4.1 , $\bar{H} = \frac{\langle \tilde{0} | H | \tilde{0} \rangle}{\langle \tilde{0} | \tilde{0} \rangle} = \frac{\int_{-\infty}^{\infty} \langle \tilde{0} | H | x \rangle \langle x | \tilde{0} \rangle dx}{\int_{-\infty}^{\infty} \langle \tilde{0} | x \rangle \langle x | \tilde{0} \rangle dx}$

For the harmonic oscillator:

$$H = \frac{P^2}{2m} + \frac{1}{2} m\omega^2 x^2 = \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2} m\omega^2 x^2$$

we are told to use the text function:

$$\langle x | \tilde{0} \rangle = e^{-\beta |x|}$$

$$\therefore \bar{H} = \frac{\int_{-\infty}^{\infty} \frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} e^{-\beta|x|}\right) e^{-\beta|x|} dx + \int_{-\infty}^{\infty} \frac{1}{2} m\omega^2 x^2 e^{-\beta 2|x|} dx}{\int_{-\infty}^{\infty} e^{-2\beta|x|} dx}$$

$$= \frac{\frac{\beta^2 \hbar^2}{2m} \cdot 2\int_0^{\infty} e^{-2\beta x} dx + \frac{1}{2} m\omega^2 \, 2\int_0^{\infty} x^2 e^{-\beta 2x} dx}{2\int_0^{\infty} e^{-2\beta x} dx}$$

using $\int_0^{\infty} e^{-\alpha x} x^n dx = \frac{n!}{\alpha^{n+1}}$

$$\bar{H} = \frac{\frac{\beta \hbar^2}{4m} + \frac{m\omega^2}{8\beta^3}}{1/2\beta} = \frac{\beta^2 \hbar^2}{2m} + \frac{m\omega^2}{\beta^4}$$

To find $\bar{H}_{min}$, do $\frac{\partial \bar{H}}{\partial \beta} = 0 \Rightarrow \frac{\beta \hbar^2}{m} - \frac{m\omega^2}{2\beta^3} = 0 \Rightarrow \beta = \sqrt{\frac{m\omega}{\hbar}}\, 2^{-1/4}$

$$\therefore \bar{H}_{min} = \frac{\omega\hbar}{2\sqrt{2}} + \frac{\omega\hbar}{4\sqrt{2}} = \frac{3\hbar\omega}{4\sqrt{2}} = \boxed{0.5303\ \hbar\omega}$$

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That estimate of the groud state energy is very close to the true groud state energy $\frac{1}{2}\hbar\omega$. Nice.