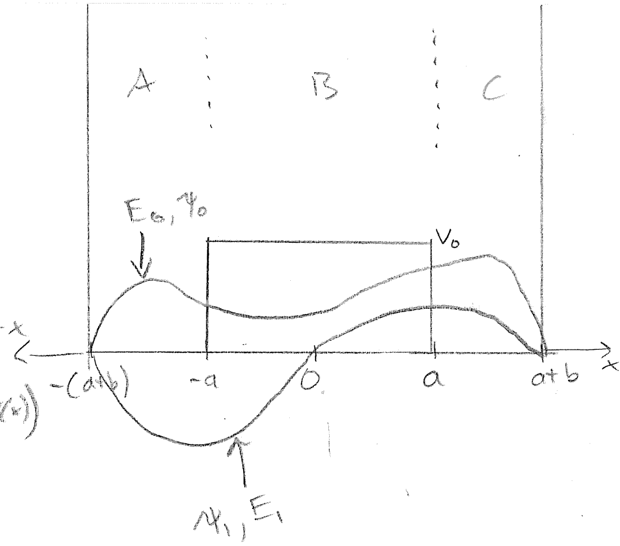
$\boxed{4.6}$ Boundary Conditions

$$\psi(-(a+b)) = \psi(a+b) = 0$$

$$\psi_A(-a) = \psi_B(-a) \qquad \frac{d}{dx}\psi_A(-a) = \frac{d}{dx}\psi_B(-a)$$

$$\psi_B(a) = \psi_C(a) \qquad \frac{d}{dx}\psi_B(a) = \frac{d}{dx}\psi_C(a)$$



Schrödinger Equation $\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) + V\psi(x) = E\psi(x)\right)$

Regions

A: $\frac{-\hbar^2}{2m}\frac{d^2}{dx^2}\psi_A(x) = E\psi_A(x)$

B: $\frac{-\hbar^2}{2m}\frac{d^2}{dx^2}\psi_B(x) + V_0\psi_B(x) = E\psi_B(x)$

C: $\frac{-\hbar^2}{2m}\frac{d^2}{dx^2}\psi_C(x) = E\psi_C(x)$

Solutions to the Schrödinger Eq in the Even case (ψ_0)

$\psi_A(x) = e^{ikx}$, but to match the boundary condition $\psi(-(a+b)) = 0$,

we choose $\psi_{0A}(x) = C_1 \sin(k(x+(a+b)))$ where $k = \sqrt{\frac{2m}{\hbar^2}E}$

also, $\psi_{0,C}(x) = -C_1\sin(k(x-(a+b))) = \psi_{0A}(-x)$ due to symmetry

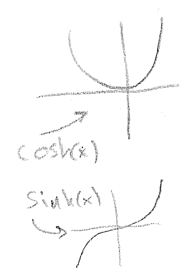
The solution for $\psi_B(x)$ is a little trickier due to the damping.

$$\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi_B(x) = (V_0 - E)\psi_B(x) \Rightarrow \psi_B''(x) - \frac{2m}{\hbar^2}(V_0 - E)\psi_B(x) = 0$$

The solutions are $\psi_B(x) = C_2\sinh(k_0 x) + C_3\cosh(k_0 x)$

where I let $k_0 = \sqrt{\frac{2m}{\hbar^2}(V_0 - E)}$

$\rightarrow$ for even parity, $\psi_B(x) = \psi_B(-x)$, so only Cosh survives.



in all

$$\psi_0 = \begin{cases} C_1\sin(k(x-(a+b))) & -(a+b) < x < -a \quad (A) \\ C_3\cosh(k_0 x) & -a < x < a \quad (B) \\ -C_1\sin(k(x+(a+b))) & a < x < a+b \quad (C) \end{cases}$$

Solutions for the Odd case (ψ_1)

Similarly,

$$\psi_1 = \begin{cases} -C_1\sin(k-(a+b)) & -(a+b) < x < -a \quad (A) \\ C_3\sinh(k_0 x) & -a < x < a \quad (B) \\ -C_1\sin(k(-x+(a+b))) & a < x < a+b \quad (C) \end{cases}$$

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Now, we match boundary conditions

we'll use the RHS of the well b/c it's easier to think about.

$$\psi_B(a) = \psi_c(a) \Rightarrow \frac{\psi_B(a)}{\psi_c(a)} = 1 \quad \text{and} \quad \psi_B'(a) = \psi_c'(a) \Rightarrow \frac{\psi_B'(a)}{\psi_c'(a)} = 1$$

$$\Rightarrow \frac{\psi_B'(a)}{\psi_B(a)} = \frac{\psi_c'(a)}{\psi_c(a)}$$

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→ Even parity (ψ_0 case)

$$\frac{k_0 C_3 \sinh(k_0 a)}{C_3 \cosh(k_0 a)} = \frac{+k C_1 \cos(k(-a+a+b))}{-C_1 \sin(k(-a+a+b))} \Rightarrow k_0 \tanh(k_0 a) = -k\cot(kb)$$

Since $V_0 >> 1$, $k_0 >> 1$ as well.

Note $\tanh(x) \approx (1-e^{-2x})$ where $x >> 1$

Thus, we have

$$k_0(1-2e^{-2k_0 a}) = -k\cot(kb)$$

Let's consider the case where $V_0 \to \infty$. Then $k_0 \to \infty$ too,

and $k_0 = -k\cot(kb)$

In this limit, $k_0 \to \infty$, but $-k\cot(kb)$ only goes to infinity at $kb = \pi$

Thus $k = \frac{\pi}{b}$

Back to the real world of $V_0 >> 1$, but finite. We'll assume $k \approx \frac{\pi - \delta}{b}$

$$k_0(1-2e^{-2k_0 a}) \approx -\frac{\pi}{b}\frac{\cos(\pi)}{\sin(\pi)} = +\frac{\pi}{b}\frac{1}{\sin(\delta-\pi)} \approx \frac{\pi}{b\delta}, \text{ where } \delta << 1.$$

$$\therefore k_0(1-2e^{-2k_0 a}) = \frac{\pi}{b\delta} \Rightarrow \delta = \frac{\pi}{k_0 b}\frac{1}{(1-2e^{-2k_0 a})}$$

$$\text{so } k \approx \frac{\pi - \frac{\pi}{k_0 b}\frac{1}{(1-2e^{-2k_0 a})}}{b} \approx \frac{\pi}{b}\left(1 - \frac{1}{k_0 b}(1+2e^{-2k_0 a})\right) \quad \text{since } k_0 >> 1.$$

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Now, noting $k = \sqrt{\frac{2mE_0}{\hbar^2}} \Rightarrow E_0 = \frac{\hbar^2 k^2}{2m} = \frac{\hbar^2\pi^2}{2mb^2}\left(1 + \frac{1}{k_0 b}(1-2e^{-2k_0 a})\right)^2$

finally, $(1-x)^2 \approx 1-2x$, when $x << 1$, and in this case $\frac{1}{k_0 b}(1+2e^{-2k_0 a}) << 1$

$$\text{so } E_0 = \frac{\hbar^2\pi^2}{2mb^2}\left(1 - \frac{2}{k_0 b}(1+2e^{-2k_0 a})\right)$$

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→ Odd Parity (ψ_1 case) This is a lot of writing. In my scratch work, I found VERY similarly that $E_1 = \frac{\hbar^2\pi^2}{2mb^2}\left(1 - \frac{2}{k_0 b}(1-2e^{-2k_0 a})\right)$

minus sign b/c $\coth(k_0 a) \approx 1 + 2e^{-2k_0 a}$

$$E_0 - E_1 = \frac{\hbar^2\pi^2}{2mb^2}\left[\frac{-2}{k_0 b}\left(1 + 2e^{-2k_0 a} - 1 + 2e^{-2k_0 a}\right)\right] = \boxed{\frac{4\hbar^2\pi^2}{mb^3 k_0}e^{-2k_0 a}}$$