4.4 ⓐ I use equation 3.8.64 $\quad \mathcal{Y}_{\ell}^{j=\ell\pm\frac{1}{2},\, m} = \frac{1}{\sqrt{2\ell+1}}\begin{bmatrix} \pm\sqrt{\ell \pm m + \frac{1}{2}}\; Y_{\ell}^{m-1/2}(\theta,\phi) \\ \sqrt{\ell \mp m + \frac{1}{2}}\; Y_{\ell}^{m+1/2}(\theta,\phi) \end{bmatrix}$

In this case $\ell=0$, $j=\frac{1}{2}$, and $m=\frac{1}{2}$, so we use the upper signs, since $\ell \oplus \frac{1}{2} = j$. Let's just plug in the quantum numbers.

$$\mathcal{Y}_{\ell=0}^{j=\frac{1}{2};\, m=\frac{1}{2}} = \begin{bmatrix} +\sqrt{+1}\; Y_0^0(\theta,\phi) \\ \sqrt{0}\; Y_0^1(\theta,\phi) \end{bmatrix} = \begin{bmatrix} \sqrt{1}\,\frac{1}{2\sqrt{\pi}} \\ 0 \end{bmatrix} = \boxed{\frac{1}{\sqrt{4\pi}}\begin{bmatrix} 1 \\ 0 \end{bmatrix}}$$

ⓑ $$\vec{\sigma}\cdot\vec{x} = \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{bmatrix}\cdot\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \sigma_1 x + \sigma_2 y + \sigma_3 z = \begin{bmatrix} 0 & x \\ x & 0 \end{bmatrix} + \begin{bmatrix} 0 & -iy \\ iy & 0 \end{bmatrix} + \begin{bmatrix} z & 0 \\ 0 & -z \end{bmatrix}$$

$$= \begin{bmatrix} z & x-iy \\ x+iy & -z \end{bmatrix}$$

$$\therefore\ \vec{\sigma}\cdot\vec{x}\ \mathcal{Y}_{\ell=0}^{j=\frac{1}{2}\ m=\frac{1}{2}} = \frac{1}{\sqrt{4\pi}}\begin{bmatrix} z & x-iy \\ x+iy & -z \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{4\pi}}\begin{bmatrix} z \\ x+iy \end{bmatrix}$$

Convert to spherical coordinates so that we can jigger it into spherical harmonic form:

$$= \frac{1}{\sqrt{4\pi}}\begin{bmatrix} r\cos\theta \\ r\sin\theta(\cos\phi + i\sin\phi) \end{bmatrix} = \frac{1}{\sqrt{4\pi}}\begin{bmatrix} r\cos\theta \\ r\sin\theta\, e^{i\phi} \end{bmatrix} = r\begin{bmatrix} \frac{1}{\sqrt{4\pi}}\cos\theta \\ \frac{1}{\sqrt{4\pi}}\sin\theta\, e^{i\phi} \end{bmatrix}$$

$$= r\begin{bmatrix} \frac{1}{\sqrt{3}}Y_1^0 \\ -\sqrt{\frac{2}{3}}\,Y_1^1 \end{bmatrix} = r\,\frac{1}{\sqrt{3}}\begin{bmatrix} Y_1^0 \\ -\sqrt{2}\,Y_1^1 \end{bmatrix}$$

Now, we'll reverse engineer a spin-angular function. Note that clearly $\ell=1$, and to make the m's work, we should set $m=\frac{1}{2}$, $j=\frac{1}{2}$, and use the bottom signs.

$$= -r\,\frac{1}{\sqrt{2(1)+1}}\begin{bmatrix} -\sqrt{(1)-(\frac{1}{2})+(\frac{1}{2})}\; Y_1^{(\frac{1}{2}-\frac{1}{2})}(\theta,\phi) \\ \sqrt{(1)+(\frac{1}{2})+(\frac{1}{2})}\; Y_1^{(\frac{1}{2}+\frac{1}{2})}(\theta,\phi) \end{bmatrix} = \boxed{-|\vec{x}|\ \mathcal{Y}_{\ell=1}^{j=\frac{1}{2},\, m=\frac{1}{2}}}$$

ⓒ As we saw in class, $\vec{S}\cdot\vec{x} \propto \vec{\sigma}\cdot\vec{x}$ is a pseudoscalar, so it is odd under parity.

→ Since $\mathcal{Y}_{\ell}^{j,m} \to (-1)^{\ell}\,\mathcal{Y}_{\ell}^{j,m}$ under parity, we know that $\mathcal{Y}_{\ell=0}^{\frac{1}{2}\,\frac{1}{2}}$ is even under parity.

→ Then we hit it w/ an odd $\vec{S}\cdot\vec{x}$ operator, and get an overall result which is odd under parity.

→ Look! $-|\vec{x}|\,\mathcal{Y}_{\ell=1}^{j=\frac{1}{2},\, m=\frac{1}{2}}$ is odd under parity (ℓ=odd), so this checks out.

$\vec{S}\cdot\vec{x}$, when rotated, acts like a scalar — it is invariant.
Thus, as expected, the parity does not change when we act it on $\mathcal{Y}$.

$\frac{20}{20}$