4.10 (a) Under $\hat{\Theta}$, what happens to $\mathcal{D}(R)|j,m\rangle$?

$$\hat{\Theta}\mathcal{D}(R)|j,m\rangle = \hat{\Theta}\exp\left(\frac{-i\vec{J}\cdot\hat{n}\phi}{\hbar}\right)|jm\rangle \quad \text{by } 3.5.42$$
$$= \hat{\Theta}\exp\left(\frac{-i\vec{J}\cdot\hat{n}\phi}{\hbar}\right)\hat{\Theta}^{-1}\hat{\Theta}|jm\rangle$$
$$= \exp\left(\frac{-i\vec{J}\cdot\hat{n}\phi}{\hbar}\right)\hat{\Theta}|jm\rangle$$

because the operators changed $-i$ to $+i$, and $\vec{J}$ to $-\vec{J}$ by 4.4.53.

$$= \mathcal{D}(R)\,\hat{\Theta}|jm\rangle = (-1)^m \mathcal{D}(R)|jm\rangle \quad \text{by } 4.4.78.$$

$$\Rightarrow \boxed{\hat{\Theta}\mathcal{D}(R)|j,m\rangle = (-1)^m\mathcal{D}(R)|j,m\rangle}$$

$\frac{20}{20}$

(b) Prove: $\mathcal{D}^{(j)*}_{m'm}(R) = (-1)^{m-m'}\mathcal{D}^{(j)}_{-m',-m}(R)$

Proof: $\hat{\Theta}\mathcal{D}^{(j)}_{-m',m}(R) = \langle j\, -m'|\hat{\Theta}\mathcal{D}(R)|j\ m\rangle$

$$= \sum_n \langle j\, -m'|\hat{\Theta}|j\,n\rangle\langle j\,n|\mathcal{D}(R)|j\,m\rangle^*$$

where I inserted a complete set of states. (By removing the $\hat{\Theta}$ operator from the 2nd sum, we must complex conjugate).

$$= \sum_n (-1)^n\langle j\,-m'|j\,n\rangle\langle j\,n|\mathcal{D}(R)|j\,m\rangle^* \quad (\text{by } 4.4.78)$$
$$= \sum_n (-1)^n\delta_{-m',n}\langle j\,n|\mathcal{D}(R)|j\,m\rangle^*$$
$$= (-1)^{-m'}\langle j\,-m'|\mathcal{D}(R)|j\,m\rangle^* = (-1)^{-m'}\mathcal{D}^{(j)*}_{-m',m}(R)$$

Reindex $\rightarrow (-1)^{m'}\mathcal{D}^{(j)*}_{m',m}(R)$

Next,

$$\hat{\Theta}\mathcal{D}^{(j)}_{-m',m}(R) = \langle j\,-m'|(-1)^m\mathcal{D}(R)|j\,-m\rangle \quad (4.4.78 \text{ again}).$$
$$= (-1)^m\mathcal{D}^{(j)}_{-m',-m}$$

$$\Rightarrow (-1)^{m'}\mathcal{D}^{(j)*}_{m',m}(R) = (-1)^m\mathcal{D}^{(j)}_{-m',-m}$$
$$\Rightarrow \boxed{\mathcal{D}^{(j)*}_{m'm}(R) = (-1)^{m-m'}\mathcal{D}^{(j)}_{-m',-m}(R)} \quad \square$$

$\frac{20}{20}$

(c) Prove: $\hat{\Theta}|jm\rangle = i^{2m}|j\,-m\rangle$

Proof: $\hat{\Theta}|jm\rangle = (-1)^m|j\,-m\rangle$ by 4.4.78

Note that $-1 = i^2$ because $i = \sqrt{-1}$

Thus $(-1)^m = i^{2m}$

$$\Rightarrow \boxed{\hat{\Theta}|jm\rangle = i^{2m}|j\,-m\rangle}$$

half-integer

$\frac{20}{20}$

$\frac{100}{100}$