Sakurai 3-32

ⓐ We are to write the following combinations in terms of spherical tensors:

$\underline{xy}$: We'll need $Y_2^{+2} = \sqrt{\frac{15}{32\pi}}\frac{(x+iy)^2}{r^2}$, $Y_2^{-2} = \sqrt{\frac{15}{32\pi}}\frac{(x-iy)^2}{r^2}$ (3.11.17)

Thus, $\boxed{-\frac{1}{4}i\sqrt{\frac{32\pi}{15}}\,r^2\left(Y_2^{+2} - Y_2^{-2}\right) = xy}$ ✓ b/c I just took linear combinations until the correct answer popped out

$\underline{x^2-y^2}$: using 3.11.17 again $\boxed{x^2-y^2 = \frac{1}{2}\sqrt{\frac{32\pi}{15}}\,r^2\left(Y_2^{+2} + Y_2^{-2}\right)}$ ✓

$\underline{xz}$: One can look up $Y_2^{\pm 1}$ on wikipedia...

$Y_2^{+1} = -\frac{1}{2}\sqrt{\frac{15}{2\pi}}\frac{(x+iy)z}{r^2}$, $Y_2^{-1} = \frac{1}{2}\sqrt{\frac{15}{2\pi}}\frac{(x-iy)z}{r^2}$

$$\boxed{xz = -\sqrt{\frac{2\pi}{15}}\,r^2\left(Y_2^{+1} - Y_2^{-1}\right)}$$ ✓

ⓑ Let's evaluate Q first.

$Q = e\langle\alpha, j, m=j|(3z^2-r^2)|\alpha, j, m=j\rangle$

We need to write $3z^2-r^2$ in terms of spherical harmonics.

$Y_2^0 = \frac{1}{4}\sqrt{\frac{5}{\pi}}\frac{(2z^2-x^2-y^2)}{r^2} = \frac{1}{4}\sqrt{\frac{5}{\pi}}\frac{(3z^2-z^2-x^2-y^2)}{r^2} = \frac{1}{4}\sqrt{\frac{5}{\pi}}\frac{(3z^2-r^2)}{r^2}$

Thus, $Q = e\langle\alpha, j, m=j|4\sqrt{\frac{\pi}{5}}\,r^2\,Y_2^0|\alpha, j, m=j\rangle$

Now we use the Wigner-Eckart Theorem: (3.11.31)

$$\langle\alpha, j, m=j|T_0^{(2)}|\alpha, j, m=j\rangle = \langle j2; j0|j2; jj\rangle\frac{\langle\alpha j\|T^{(2)}\|\alpha j\rangle}{\sqrt{2j+1}}$$

$$\Rightarrow Q = e4\sqrt{\frac{\pi}{5}}\,r^2\,\langle j2\,j0|j2\,jj\rangle\frac{\langle\alpha j\|Y_2\|\alpha j\rangle}{\sqrt{2j+1}}$$

We also need to write our "target" $e\langle\alpha j m'|(x^2-y^2)|\alpha, j m=j\rangle$ in this form.

from part ⓐ we have $e\frac{1}{2}\sqrt{\frac{32\pi}{15}}\,r^2\left(\langle\alpha j m'|Y_2^{+2}|\alpha, j\, m=j\rangle + \langle\alpha j m'|Y_2^{-2}|\alpha, j m=j\rangle\right)$

By W-E-Thm: $= e\frac{1}{2}\sqrt{\frac{32\pi}{15}}\,r^2\left(\cancel{\langle j2\; j2|j2\; jm'\rangle} + \langle j2\; j(-2)|j2\; jm'\rangle\right)\frac{\langle\alpha j\|Y_2\|\alpha j\rangle}{\sqrt{2j+1}}$

0 since $m' \neq j+2$

Plug in for $\frac{\langle\alpha j\|Y_2\|\alpha j\rangle}{\sqrt{2j+1}} = \frac{\sqrt{5}Q}{e4\sqrt{\pi}\,r^2\langle j2\,j0|j2\,jj\rangle}$

$$\boxed{e\langle\alpha j m'|(x^2-y^2)|\alpha, j, m=j\rangle = \frac{Q}{\sqrt{6}}\frac{\langle j2\; j(-2)|j2\; jm'\rangle}{\langle j2\; j0|j2\; jj\rangle}}$$ ✓

⑩