Sakurai 3.30

ⓐ $\vec{U} = (U_x, U_y, U_z)$, $\vec{V} = (V_x, V_y, V_z)$

we will employ theorem 3.1 which says that if $X_{q_1}^{(k_1)}$ and $Z_{q_2}^{(k_2)}$ are irreducible spherical tensors, then so is $T_q^{(k)} = \sum_{q_1}\sum_{q_2} \langle k_1 k_2; q_1 q_2 | k_1 k_2; qk\rangle X_{q_1}^{(k_1)} Z_{q_2}^{(k_2)}$

Here we are given $k_1 = k_2 = 1$ and let $X \to U$, $Z \to V$

q, q_1, and q_2 are allowed to range from $-1, 0, 1$ only.

→ when $q = -1$, then $qk = -1$, so $q_1 + q_2 = -1$ as well.

oops... this is the hard way...

The options are $q_1 = 0$, $q_2 = -1$, or $q_2 = 0$, $q_1 = -1$.

The clebsch gordon coefficient can be looked up.

For $j_1 = j_2 = k_1 = k_2 = 1$, $m = k = 1$, and $m_1 = q_1 = 0$, $m_2 = q_2 = \pm 1$ we have $+\sqrt{\frac{1}{2}}$

For $m_1 = q_1 = \pm 1$, and $m_2 = q_2 = 0$, we have $-\sqrt{\frac{1}{2}}$

Thus, $T_{-1}^{(1)} = \left(+\frac{1}{\sqrt{2}} U_0^{(1)} V_{-1}^{(1)} - \frac{1}{\sqrt{2}} U_1^{(1)} V_0^{(1)}\right)$ ①

→ when $q = 0$, we can have $q_1 = +1$, $q_2 = -1$, or $q_1 = -1$, $q_2 = +1$.

$T_0^{(1)} = +\sqrt{\frac{1}{2}} U_{+1}^{(1)} V_{-1}^{(1)} - \sqrt{\frac{1}{2}} U_{-1}^{(1)} V_{+1}^{(1)}$ ②

→ similarly:

$T_1^{(1)} = +\sqrt{\frac{1}{2}} U_1 V_0 - \sqrt{\frac{1}{2}} U_0 V_1$ ③

Now, as in the previous problem, $U_{\pm 1} = \mp \frac{W_x \pm i W_y}{\sqrt{2}}$, $U_0 = U_z$ (same goes for V)

Plug those into ①, ② and ③ (see mathematica for algebra)

$$T_q^{(1)} = \frac{1}{2}\begin{bmatrix} V_z (U_x - iU_y) + U_z (V_x + iV_y) \\ -i\sqrt{2}\,(U_y V_x - U_x V_y) \\ V_z (U_x - iU_y) - U_z (V_x - iV_y) \end{bmatrix}$$ ✓

ⓑ This time use 3.11.26.

$$T_0^{(2)} = \frac{U_{+1} V_{-1} + 2 U_0 V_0 + U_{-1} V_{+1}}{\sqrt{6}}$$

$$T_{\pm 1}^{(2)} = \frac{U_{\pm 1} V_0 + U_0 V_{\pm 1}}{\sqrt{2}}, \quad T_{\pm 2}^{(2)} = U_{\pm 1} V_{\pm 1}$$

Plug in those same definitions of $U_\pm$, $V_\pm$, U_0, and V_0 from part a, the previous problem, and beyond. (See Mathematica).

See Mathematica Output for the solutions. ✓ 10

part (a)

Here are the definitions of the spherical tensors U and V:

In[89]:=
$$\texttt{U} = \left\{-\frac{(\texttt{Ux} + i\,\texttt{Uy})}{\sqrt{2}},\ \texttt{Uz},\ \frac{(\texttt{Ux} - i\,\texttt{Uy})}{\sqrt{2}}\right\};$$
$$\texttt{V} = \left\{-\frac{(\texttt{Vx} + i\,\texttt{Vy})}{\sqrt{2}},\ \texttt{Vz},\ \frac{(\texttt{Vx} - i\,\texttt{Vy})}{\sqrt{2}}\right\};$$

Now I plug the elements of U and V in terms of cartesian coordinates into my three expressions. Note that *Mathematica*'s indexing using 1, 2, and 3, while I really mean -1, 0, and 1. Understand as:
$-1 \longrightarrow 1 \qquad 0 \longrightarrow 2 \qquad +1 \longrightarrow 3$

In[106]:=
$$\frac{1}{\sqrt{2}}\texttt{U[[2]]} * \texttt{V[[1]]} + \frac{-1}{\sqrt{2}}\texttt{U[[3]] V[[2]]}$$
$$\frac{1}{\sqrt{2}}\texttt{U[[3]]} * \texttt{V[[1]]} - \frac{1}{\sqrt{2}}\texttt{U[[1]] V[[3]]}\ \texttt{// FullSimplify}$$
$$\frac{1}{\sqrt{2}}\texttt{U[[3]]} * \texttt{V[[2]]} - \frac{1}{\sqrt{2}}\texttt{U[[2]] V[[3]]}$$

Out[106]=
$$\frac{1}{2}\,\mathrm{Uz}\,(\mathrm{Vx} + i\,\mathrm{Vy}) + \frac{1}{2}\,(\mathrm{Ux} - i\,\mathrm{Uy})\,\mathrm{Vz}$$

Out[107]=
$$\frac{i\,(\mathrm{Uy}\,\mathrm{Vx} - \mathrm{Ux}\,\mathrm{Vy})}{\sqrt{2}}$$

Out[108]=
$$\frac{1}{2}\,\mathrm{Uz}\,(\mathrm{Vx} - i\,\mathrm{Vy}) - \frac{1}{2}\,(\mathrm{Ux} - i\,\mathrm{Uy})\,\mathrm{Vz}$$

part (b)

As in the previous part, let's plug the components of **V** and **U** into the 5 components of the spherical tensor of rank 2.

In[115]:=
$$(*\ \mathbf{T}^{(2)}_{-2}\ *)\ \texttt{U[[1]] V[[1]] // FullSimplify}$$

Out[115]=
$$\frac{1}{2}\,(\mathrm{Ux} + i\,\mathrm{Uy})\,(\mathrm{Vx} + i\,\mathrm{Vy})$$

In[114]:=
$$(*\ \mathbf{T}^{(2)}_{-1}\ *)\ \frac{\texttt{U[[1]] V[[2]]} + \texttt{U[[2]] V[[1]]}}{\sqrt{2}}\ \texttt{// FullSimplify}$$

Out[114]=
$$\frac{1}{2}\,(\mathrm{Uz}\,(\mathrm{Vx} + i\,\mathrm{Vy}) + (\mathrm{Ux} + i\,\mathrm{Uy})\,\mathrm{Vz})$$

In[111]:=
$$(*\ \mathbf{T}^{(2)}_{0}\ *)\ \frac{\texttt{U[[3]] V[[1]]} + 2\,\texttt{U[[2]] V[[2]]} + \texttt{U[[1]] V[[3]]}}{\sqrt{6}}\ \texttt{// FullSimplify}$$

Out[111]=
$$-\frac{\mathrm{Ux}\,\mathrm{Vx} + \mathrm{Uy}\,\mathrm{Vy} - 2\,\mathrm{Uz}\,\mathrm{Vz}}{\sqrt{6}}$$

In[113]:=
$$(*\ \mathbf{T}^{(2)}_{+1}\ *)\ \frac{\texttt{U[[3]] V[[2]]} + \texttt{U[[2]] V[[3]]}}{\sqrt{2}}\ \texttt{// FullSimplify}$$

Out[113]=
$$\frac{1}{2}\,(-\mathrm{Uz}\,\mathrm{Vx} + i\,\mathrm{Uz}\,\mathrm{Vy} - \mathrm{Ux}\,\mathrm{Vz} + i\,\mathrm{Uy}\,\mathrm{Vz})$$

In[119]:=
$$(*\ \mathbf{T}^{(2)}_{+2}\ *)\ \texttt{U[[3]] V[[3]] // FullSimplify}$$

Out[119]=
$$\frac{1}{2}\,(\mathrm{Ux} - i\,\mathrm{Uy})\,(\mathrm{Vx} - i\,\mathrm{Vy})$$