$\boxed{\text{Sakurai 3-3}}$ $U = \frac{a_0 + i\sigma\cdot\vec{a}}{a_0 - i\sigma\cdot\vec{a}} = \frac{a_0 I + i\sigma_1 a_1 + i\sigma_2 a_2 + i\sigma_3 a_3}{a_0 I - i\sigma_1 a_1 - i\sigma_2 a_3 - i\sigma_3 a_3}$

(a) using $\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$, $\sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$$U = \begin{bmatrix} a_0 + ia_3 & ia_1 + a_2 \\ ia_1 - a_2 & a_0 - ia_3 \end{bmatrix} \begin{bmatrix} a_0 - ia_3 & -ia_1 - a_2 \\ -ia_1 + a_2 & a_0 + ia_3 \end{bmatrix}^{-1}$$

Do the inversion in Mathematica.

$$U = \begin{bmatrix} a_0 + ia_3 & ia_1 + a_2 \\ ia_1 - a_2 & a_0 - ia_3 \end{bmatrix} \begin{bmatrix} a_0 + ia_3 & ia_1 + a_2 \\ ia_1 - a_2 & a_0 - ia_3 \end{bmatrix} \frac{1}{a_0^2 + a_1^2 + a_2^2 + a_3^2}$$

$$U = \frac{1}{a_0^2 + a_1^2 + a_2^2 + a_3^2} \begin{bmatrix} (a_0 + ia_3)(a_0 + ia_3) + (ia_1 + a_2)(ia_1 - a_2) & (a_0 + ia_3)(ia_1 + a_2) + (a_0 - ia_3)(ia_1 + a_2) \\ (a_0 + ia_3)(-ia_1 - a_2) + (ia_1 - a_2)(a_0 - ia_3) & (ia_1 + a_2)(ia_1 - a_2) + (a_0 - ia_3)(a_0 - ia_3) \end{bmatrix}$$

$$U = \frac{1}{a_0^2 + a_1^2 + a_2^2 + a_3^2} \begin{bmatrix} (a_0 + ia_3)^2 - a_1^2 - a_2^2 & 2a_0(ia_1 + a_2) \\ 2ia_0(a_1 + ia_2) & (a_0 - ia_3)^2 - a_1^2 - a_2^2 \end{bmatrix}$$

Unimodular: We must show the determinant of U is ± 1. I'm not sure if there is a trick to do this quickly - this is a lot of math to do by hand. In Mathematica, I took $\boxed{\text{Det}(U) = 1}$

Unitary: We now show that $UU^{+} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$U^{+} = \frac{1}{a_0^2 + a_1^2 + a_2^2 + a_3^2} \begin{bmatrix} (a_0 - ia_3)^2 - a_1^2 - a_2^2 & -2ia_0(a_1 - ia_2) \\ 2a_0(-ia_1 + a_2) & (a_0 - ia_3)^2 - a_1^2 - a_2^2 \end{bmatrix}$$

$\boxed{UU^{+} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}$ ✓ Once again, this would take a lot of multiplication, and I decided against it / went w/ MMA instead.

(b) We can think of $U = \frac{1}{a_0^2 + a_1^2 + a_2^2 + a_3^2} \begin{bmatrix} (a_0^2 - a_1^2 - a_2^2 - a_3^2) + i2a_0 a_3 & 2a_0 ia_1 + 2a_0 a_2 \\ 2ia_0 a_1 - 2a_0 a_2 & (a_0^2 - a_1^2 - a_2^2 - a_3^2) - i2a_0 a_3 \end{bmatrix}$

in the form $U(a,b) = \begin{bmatrix} a & b \\ -b^* & a^* \end{bmatrix}$ as in 3.3.7, where

$$a = \frac{a_0^2 - a_1^2 - a_2^2 - a_3^2 + i2a_0 a_3}{a_0^2 + a_1^2 + a_2^2 + a_3^2}, \text{ and } b = \frac{2a_0 ia_1 + 2a_0 a_2}{a_0^2 + a_1^2 + a_2^2 + a_3^2}$$

Now, from 3.3.10, $\text{Re}(a) = \cos(\phi/2)$, $\text{Re}(b) = -n_y \sin(\phi/2)$, $\text{Im}(a) = -n_z \sin(\phi/2)$, $\text{Im}(b) = -n_x \sin(\phi/2)$

$\to \text{Re}(a) = \frac{a_0^2 - a_1^2 - a_2^2 - a_3^2}{a_0^2 + a_1^2 + a_2^2 + a_3^2} \Rightarrow \boxed{2\cos^{-1}\left(\frac{a_0^2 - a_1^2 - a_2^2 - a_3^2}{a_0^2 + a_1^2 + a_2^2 + a_3^2}\right) = \phi}$ Let $|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$, then ✓ $\boxed{\phi = 2\cos^{-1}\left(\frac{a_0^2 - |\vec{a}|^2}{a_0^2 + |\vec{a}|^2}\right)}$

$\to \text{Re}(b) = -n_y \sin(\phi/2) = -n_y \sin\left(\cos^{-1}\left(\frac{a_0^2 - |\vec{a}|^2}{a_0^2 + |\vec{a}|^2}\right)\right) = -n_y\sqrt{1 - \frac{a_0^2 - |\vec{a}|^2}{a_0^2 + |\vec{a}|^2}} \Rightarrow$ By Mathematica $\Rightarrow n_y = \frac{-a_2}{|\vec{a}|}$

similarly, $n_x = \frac{-a_1}{|\vec{a}|}$, $n_z = \frac{-a_3}{|\vec{a}|}$, so $\boxed{\hat{n} = \frac{-1}{|\vec{a}|}(a_1\hat{x} + a_2\hat{y} + a_3\hat{z})}$ ✓

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#1 Sakurai Problem 3.3a

In[332]:=
```
numerator = {{a0 + i * a3, i * a1 + a2}, {i * a1 - a2, a0 - i * a3}};
denominator = Inverse[{{a0 - i * a3, -i * a1 - a2}, {-i * a1 + a2, a0 + i * a3}}];
```

In[334]:=
```
U = numerator.denominator // FullSimplify
```

Out[334]=

$$\left\{\left\{-1+\frac{2\,a0\,(a0+i\,a3)}{a0^2+a1^2+a2^2+a3^2},\ \frac{2\,a0\,(i\,a1+a2)}{a0^2+a1^2+a2^2+a3^2}\right\},\left\{\frac{2\,i\,a0\,(a1+i\,a2)}{a0^2+a1^2+a2^2+a3^2},\ -1+\frac{2\,a0\,(a0-i\,a3)}{a0^2+a1^2+a2^2+a3^2}\right\}\right\}$$

In[339]:=
```
U * (a0^2 + a1^2 + a2^2 + a3^2) // MatrixForm // FullSimplify
```

Out[339]//MatrixForm=

$$\begin{pmatrix} -a1^2-a2^2+(a0+i\,a3)^2 & 2\,a0\,(i\,a1+a2) \\ 2\,i\,a0\,(a1+i\,a2) & -a1^2-a2^2+(a0-i\,a3)^2 \end{pmatrix}$$

Prove Unimodular

In[340]:=
```
Det[U] // FullSimplify
```

Out[340]= 1

Prove Unitary (I defined the $Conjugate function to stop assuming constants to be complex by default).

In[360]:=
```
$Conjugate[x_] := x /. Complex[a_, b_] :> a - I b;
```

In[364]:=
```
Udagger = FullSimplify[Transpose[$Conjugate[U]]];
```

In[363]:=
```
U.Udagger // FullSimplify // MatrixForm
```

Out[363]//MatrixForm=

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

3.3b

Here we define the Cayley-Klein parameters a and b. (a is not to be confused with the vector $\vec{a}$)

In[410]:=

$$a = \frac{(a0 + i\,a3)^2 - a1^2 - a2^2}{a0^2 + a1^2 + a2^2 + a3^2};\quad b = \frac{2\,a0\,a1\,i + 2\,a0\,a2}{a0^2 + a1^2 + a2^2 + a3^2};$$

Alternative representation to make it clear what Im(a) and Re(a) are.

In[411]:=

$$\text{Expand}\left[\frac{(a0 + i\,a3)^2 - a1^2 - a2^2}{a0^2 + a1^2 + a2^2 + a3^2}\right] \text{ // Simplify}$$

Out[411]=

$$-\frac{-a0^2 + a1^2 + a2^2 - 2\,i\,a0\,a3 + a3^2}{a0^2 + a1^2 + a2^2 + a3^2}$$

Now solve for n_x, n_y, n_z using Equations 3.3.10:

In[412]:= **Evaluate**$\left[\left\{\frac{-\mathbf{ComplexExpand[Im[b]]}}{\sqrt{1-\mathbf{ComplexExpand[Re[a]]}^2}}\right.\right.$ **// FullSimplify,**

$\frac{-\mathbf{ComplexExpand[Re[b]]}}{\sqrt{1-\mathbf{ComplexExpand[Re[a]]}^2}}$ **// FullSimplify,**

$\left.\left.\frac{-\mathbf{ComplexExpand[Im[a]]}}{\sqrt{1-\mathbf{ComplexExpand[Re[a]]}^2}}\text{ // FullSimplify}\right\}\right]$

Out[412]= $$\left\{-\frac{\text{a0 a1}}{\sqrt{\frac{\text{a0}^2\,(\text{a1}^2+\text{a2}^2+\text{a3}^2)}{(\text{a0}^2+\text{a1}^2+\text{a2}^2+\text{a3}^2)^2}}\,\left(\text{a0}^2+\text{a1}^2+\text{a2}^2+\text{a3}^2\right)},\right.$$

$$\left.-\frac{\text{a0 a2}}{\sqrt{\frac{\text{a0}^2\,(\text{a1}^2+\text{a2}^2+\text{a3}^2)}{(\text{a0}^2+\text{a1}^2+\text{a2}^2+\text{a3}^2)^2}}\,\left(\text{a0}^2+\text{a1}^2+\text{a2}^2+\text{a3}^2\right)},-\frac{\text{a0 a3}}{\sqrt{\frac{\text{a0}^2\,(\text{a1}^2+\text{a2}^2+\text{a3}^2)}{(\text{a0}^2+\text{a1}^2+\text{a2}^2+\text{a3}^2)^2}}\,\left(\text{a0}^2+\text{a1}^2+\text{a2}^2+\text{a3}^2\right)}\right\}$$

These need some simplifaction, but you get the idea!