Sakurai 3-24

For starters, the ONLY way to get $m=2$ is for $m_1 \to +$, $m_2 \to +$.

① $\boxed{|j=2, m=2\rangle = |++\rangle}$ ✓ ← already normalized

Similarly, the $m=-2$ state is also necessarily easy:

② $\boxed{|j=2, m=-2\rangle = |--\rangle}$ ✓

Ok... That was fun. Now use lowering ladder operators to lower the $|j=2, m=2\rangle$ state to go to the rest of the $j=2$ states.

→ we'll use $J_-|jm\rangle = \hbar\sqrt{(j+m)(j-m+1)}\,|j, m-1\rangle$

$J_-|j=2, m=2\rangle = (J_{1-} + J_{2-})|++\rangle$

$2\hbar|2,1\rangle = \hbar\sqrt{(1+1)(1-1+1)}\,|0+\rangle + \hbar\sqrt{(1+1)(1-1+1)}\,|+0\rangle$

③ $\Rightarrow \boxed{|j=2, m=1\rangle = \frac{1}{\sqrt{2}}|0+\rangle + \frac{1}{\sqrt{2}}|+0\rangle}$ ✓

$J_-|j=2, m=1\rangle = (J_{1-} + J_{2-})\frac{1}{\sqrt{2}}|0+\rangle + (J_{1-} + J_{2-})\frac{1}{\sqrt{2}}|+0\rangle$

$\hbar\sqrt{6}\,|2,0\rangle = \frac{\hbar}{\sqrt{2}}\sqrt{2}\,|-+\rangle + \frac{\hbar}{\sqrt{2}}\sqrt{2}\,|00\rangle + \frac{\hbar}{2}\sqrt{2}\,|00\rangle + \frac{\hbar}{2}\sqrt{2}\,|+-\rangle$

④ $\Rightarrow \boxed{|j=2, m=0\rangle = \frac{1}{\sqrt{6}}\left(|-+\rangle + |+-\rangle + 2|00\rangle\right)}$ ✓

Lower once more.

$J_-|j=2, m=0\rangle = (J_{1-} + J_{2-})\frac{1}{\sqrt{6}}\left(|-+\rangle + |+-\rangle + 2|00\rangle\right)$ similarly...

⑤ $\Rightarrow \boxed{|j=2, m=-1\rangle = \frac{1}{\sqrt{2}}\left(|-0\rangle + |0-\rangle\right)}$ ✓

Now, note $|j=1, m=1\rangle = A|+0\rangle + B|0+\rangle$, where $|A|^2 + |B|^2 = 1$

also note $\langle 2,1|1,1\rangle = 0$ (by orthogonality) $\Rightarrow \frac{1}{\sqrt{2}}(A+B) = 0$

Solve these two equations for A, B

$\frac{A}{\sqrt{2}} + \frac{B}{\sqrt{2}} = 0 \Rightarrow A = -B$, so $A = \frac{1}{\sqrt{2}}$, $B = \frac{1}{\sqrt{2}}$

⑥ $\Rightarrow \boxed{|j=1, m=1\rangle = \frac{1}{\sqrt{2}}\left(|+0\rangle - |0+\rangle\right)}$ ✓

Solution continues on the next page.

Use the lowering operator to get ~~the~~ the other two $j=1$ states.

$$J_-|j=1, m=1\rangle = (J_{1-} + J_{2-})\frac{1}{\sqrt{2}}(|+0\rangle - |0+\rangle)$$

$$|j=1, m=0\rangle = \frac{1}{\sqrt{2}}(\cancel{|00\rangle} + |+-\rangle - |-+\rangle - \cancel{|00\rangle})$$

(7) $$\boxed{|j=1, m=0\rangle = \frac{1}{\sqrt{2}}(|+-\rangle - |-+\rangle)}$$ ✓

One last time...

$$J_-|j=1, m=0\rangle = (J_{1-} + J_{2-})\frac{1}{\sqrt{2}}(|+-\rangle - |-+\rangle)$$

$$|j=1, m=-1\rangle = \frac{1}{2}(\sqrt{2}|0-\rangle - \sqrt{2}|-0\rangle)$$

(8) $$\boxed{|j=1, m=-1\rangle = \frac{1}{\sqrt{2}}(|0-\rangle - |-0\rangle)}$$ ✓

All we know about the $|j=0, m=0\rangle$ state is that it is orthogonal to all the other eight states, and that it is a linear combination of the $|+-\rangle$, $|-+\rangle$, and $|00\rangle$ states so that $m = m_1 + m_2$.

Thus

$$|j=0, m=0\rangle = A|+-\rangle + B|-+\rangle + C|00\rangle \quad \text{with } |A|^2 + |B|^2 + |C|^2 = 1$$

We know by orthogonality:

$$\langle 2,0|00\rangle = 0 = \frac{A}{\sqrt{6}} + \frac{B}{\sqrt{6}} + \frac{2C}{\sqrt{6}}$$

$$\langle 1,0|00\rangle = 0 = \frac{A}{\sqrt{2}} - \frac{B}{\sqrt{2}} = 0$$

Solve those three equations, and get that

$$A = B = \frac{1}{\sqrt{3}}, \quad C = -\frac{1}{\sqrt{3}}$$

✓

so

$$\boxed{|j=0, m=0\rangle = \frac{1}{\sqrt{3}}(|+-\rangle + |-+\rangle - |00\rangle)}$$ (10)