Sakurai 3-18

The goal is to find $\langle L_x \rangle$, $\langle L_y \rangle$, $\langle L_x^2 \rangle$, and $\langle L_y^2 \rangle$

It will be useful to write L_x, L_y in terms of L_+, L_-

Given $L_+ = L_x + iL_y$ and $L_- = L_x - iL_y$

we have $iL_y = L_+ - L_x$ and $L_x = L_- - iL_y$

Solve simultaneously for L_x, L_y, and get...

$$L_y = \frac{L_+ - L_y}{2i}, \quad L_x = \frac{L_+ + L_-}{2}$$

define these constants

Also Given: $L_+|l,m\rangle = \sqrt{(l-m)(l+m+1)}\,\hbar\,|l,m+1\rangle \equiv \mathcal{J}_+ \hbar |l,m+1\rangle$

$L_-|l,m\rangle = \sqrt{(l+m)(l-m+1)}\,\hbar\,|l,m-1\rangle \equiv \mathcal{J}_- \hbar |l,m-1\rangle$

Now we're ready....

$$\langle L_x \rangle = \langle lm|L_x|lm\rangle = \frac{1}{2}\langle lm|L_+|lm\rangle + \frac{1}{2}\langle lm|L_-|lm\rangle$$

$$= \frac{1}{2}\hbar \mathcal{J}_+ \langle lm|l\,m+1\rangle + \frac{1}{2}\hbar \mathcal{J}_- \langle lm|l\,m-1\rangle$$

$$= \frac{1}{2}\hbar(\mathcal{J}_+ \cancel{\delta_{m,m+1}} + \mathcal{J}_- \cancel{\delta_{m,m-1}}) \boxed{= 0}$$

$$\langle L_y \rangle = \langle lm|L_y|lm\rangle = \frac{1}{2i}\langle lm|L_+|lm\rangle - \frac{1}{2i}\langle lm|L_-|lm\rangle$$

$$= \frac{1}{2i}\hbar \mathcal{J}_+ \cancel{\delta_{m,m+1}} - \frac{1}{2i}\hbar \mathcal{J}_- \cancel{\delta_{m,m-1}} \boxed{= 0}$$

Note $L_x^2 = \left(\frac{L_+ + L_-}{2}\right)\left(\frac{L_+ + L_-}{2}\right) = \frac{1}{4}\left(L_+^2 + L_-^2 + L_+L_- + L_-L_+\right)$

$L_y^2 = \frac{1}{4}(L_+ - L_-)(L_+ - L_-) = \frac{1}{4}\left(L_+^2 + L_-^2 - L_+L_- - L_-L_+\right)$

$$\langle L_x^2 \rangle = \langle lm|L_x^2|lm\rangle = \frac{1}{4}|\mathcal{J}_+|^2\hbar^2 + \frac{1}{4}|\mathcal{J}_-|^2\hbar^2 - \frac{1}{4}\hbar^2 \mathcal{J}_+^* \mathcal{J}_- \cancel{\delta_{m+1,m-1}} - \frac{\hbar^2}{4}\mathcal{J}_-^* \mathcal{J}_+ \cancel{\delta_{m-1,m+1}}$$

$$= \frac{\hbar^2}{4}\left((l-m)(l+m+1) + (l+m)(l-m+1)\right)$$

$$= \frac{\hbar^2}{4}\left(l^2 + lm + l - ml - m^2 - m + l^2 - m^2 - lm + lm + l + m\right)$$

$$= \frac{\hbar^2}{4}(2l^2 + 2l - 2m^2) = \boxed{\frac{[l(l+1)\hbar^2 - m^2\hbar^2]}{2}}$$

$$\langle L_y^2 \rangle = \langle lm|L_y^2|lm\rangle = \frac{1}{4}|\mathcal{J}_+|^2\hbar^2 + \frac{1}{4}|\mathcal{J}_-|^2\hbar^2 - \frac{1}{4}\hbar^2 \mathcal{J}_+^* \mathcal{J}_- \cancel{\delta_{m+1,m-1}} - \frac{1}{4}\hbar^2 \mathcal{J}_-^* \mathcal{J}_+ \cancel{\delta_{m-1,m+1}}$$

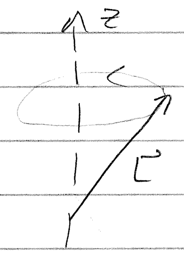
which is the same as before, so

$$\langle L_y^2 \rangle = \langle L_x^2 \rangle = [l(l+1)\hbar^2 - m^2\hbar^2$$

Solution continues on next page.

Semiclassical interpretation

Imagine an angular momentum vector $\vec{L}$ precessing like a heavy symmetrical top around the z-axis.



Clearly the L_x and L_y components of $\vec{L}$ oscillate sinusoidally, so $\langle L_x \rangle = \langle L_y \rangle = 0$.

On the other hand, L_x^2 and L_y^2 will look like $\sin^2$ and $\cos^2$, which have positive (nonzero) expectations just like we do in the quantum case.

We can also see this semi-classical correspondence using

$L^2 = L_x L_x + L_y L_y + L_z L_z$

$\langle lm | L^2 | lm \rangle = \langle lm | L_x L_x | lm \rangle + \langle lm | L_y L_y | lm \rangle + \langle lm | L_z L_z | lm \rangle$

$\hbar^2 l(l+1) = \langle L_x^2 \rangle + \langle L_y^2 \rangle + m^2 \hbar^2$

$\langle L_x^2 \rangle + \langle L_y^2 \rangle = l(l+1)\hbar^2 - m^2\hbar^2$

Then assuming symmetry btwn x and y, $\langle L_x^2 \rangle = \langle L_y^2 \rangle$

$$\Rightarrow \boxed{\langle L_x^2 \rangle = \langle L_y^2 \rangle = \frac{l(l+1)\hbar^2 - m^2\hbar^2}{2}}$$ ✓

Just as before.

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