Sakurai 3-14

Given the definition in the problem statement, we have

$$G_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i\hbar \\ 0 & i\hbar & 0 \end{bmatrix} \quad G_2 = \begin{bmatrix} 0 & 0 & i\hbar \\ 0 & 0 & 0 \\ -i\hbar & 0 & 0 \end{bmatrix} \quad G_3 = \begin{bmatrix} 0 & -i\hbar & 0 \\ i\hbar & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

I did the commutators in Mathematica. See attached.

$[G_1, G_2] = iG_3 = +i\hbar\epsilon_{123}G_3$ since $\epsilon_{123} = +1$

$[G_1, G_3] = -iG_2 = +i\hbar\epsilon_{132}G_2$ since $\epsilon_{132} = -1$

$[G_2, G_3] = iG_1 = +i\hbar\epsilon_{231}G_1$ since $\epsilon_{231} = +1$

$[G_3, G_2] = -iG_1 = +i\hbar\epsilon_{321}G_1$ since $\epsilon_{321} = -1$

$[G_2, G_1] = -iG_3 = +i\hbar\epsilon_{213}G_3$ since $\epsilon_{213} = -1$

$[G_3, G_1] = iG_2 = +i\hbar\epsilon_{312}G_2$ since $\epsilon_{312} = +1$

Thus, $[G_i, G_j] = i\hbar\epsilon_{ijk}G_k$ which is the same ✓
angular commutation relation as $[J_i, J_j] = i\hbar\epsilon_{ijk}J_k$ (eq. 3.6.2)

Now we look for some matrix U such that it transforms the G_3 matrix to J_3 in the usual manner ; $J_3 = U^+G_3U$

In mathematica, we see that the eigenvectors (which I normalized) and corresponding eigenvalues are:

$\lambda_1 = \hbar$, $\lambda_2 = -\hbar$, $\lambda_3 = 0$

$$\vec{\varepsilon}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} -i \\ 1 \\ 0 \end{bmatrix} \quad \vec{\varepsilon}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} i \\ 1 \\ 0 \end{bmatrix} \quad \vec{\varepsilon}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Thus

$$U = \begin{bmatrix} -i/\sqrt{2} & i/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow U^+ = \begin{bmatrix} -i/\sqrt{2} & 1/\sqrt{2} & 0 \\ i/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$U^+G_3U = \begin{bmatrix} -i/\sqrt{2} & 1/\sqrt{2} & 0 \\ i/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & -i\hbar & 0 \\ i\hbar & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} i/\sqrt{2} & -i/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \hbar\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ ✓

which is J_3! U works for all three.

U diagonalizes G. **What is the physical/geometric significance of this matrix?** (9)

In[31]:=
```
G1 = ⅈ ℏ {{0, 0, 0}, {0, 0, -1}, {0, 1, 0}};
G2 = ⅈ ℏ {{0, 0, 1}, {0, 0, 0}, {-1, 0, 0}};
G3 = ⅈ ℏ {{0, -1, 0}, {1, 0, 0}, {0, 0, 0}};
```

In[34]:=
```
Commutator[Ga_, Gb_] = Ga.Gb - Gb.Ga
```

Out[34]= Ga.Gb - Gb.Ga

```
Commutator[G1, G2] // MatrixForm
```

Out[35]//MatrixForm=

$$\begin{pmatrix} 0 & \hbar^2 & 0 \\ -\hbar^2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

In[36]:=
```
Commutator[G1, G3] // MatrixForm
```

Out[36]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & \hbar^2 \\ 0 & 0 & 0 \\ -\hbar^2 & 0 & 0 \end{pmatrix}$$

In[37]:=
```
Commutator[G2, G3] // MatrixForm
```

Out[37]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \hbar^2 \\ 0 & -\hbar^2 & 0 \end{pmatrix}$$

In[38]:=
```
Commutator[G3, G2] // MatrixForm
```

Out[38]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\hbar^2 \\ 0 & \hbar^2 & 0 \end{pmatrix}$$

In[39]:=
```
Commutator[G2, G1] // MatrixForm
```

Out[39]//MatrixForm=

$$\begin{pmatrix} 0 & -\hbar^2 & 0 \\ \hbar^2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

In[40]:=
```
Commutator[G3, G1] // MatrixForm
```

Out[40]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & -\hbar^2 \\ 0 & 0 & 0 \\ \hbar^2 & 0 & 0 \end{pmatrix}$$

In[42]:=
```
Eigensystem[G3] // MatrixForm
```

Out[42]//MatrixForm=

$$\begin{pmatrix} -\hbar & \hbar & 0 \\ \{i, 1, 0\} & \{-i, 1, 0\} & \{0, 0, 1\} \end{pmatrix}$$

In[60]:=
$$U = \left\{\left\{-i \big/ \sqrt{2},\ i \big/ \sqrt{2},\ 0\right\},\ \left\{1 \big/ \sqrt{2},\ 1 \big/ \sqrt{2},\ 0\right\},\ \{0, 0, 1\}\right\};$$

In[58]:=
```
Conjugate[Transpose[U]].G3.U // MatrixForm
```

Out[58]//MatrixForm=

$$\begin{pmatrix} \hbar & 0 & 0 \\ 0 & -\hbar & 0 \\ 0 & 0 & 0 \end{pmatrix}$$