$\boxed{\text{Sakurai 3-11}}$

(a) By definition, $\rho = \sum_i w_i |\alpha^{(i)}\rangle\langle\alpha^{(i)}|$ (3.4.8).

Thus $\rho(t) = \sum_i w_i |\alpha^{(i)}(t)\rangle\langle\alpha^{(i)}(t)|$ (note w_i doesn't change w/ time).

→ Fact: $\mathcal{U}(t,t_0)|\alpha^{(i)}(t_0)\rangle = |\alpha^{(i)}(t)\rangle$ (that's what time evolution is).

Thus: $\langle\alpha^{(i)}(t_0)|\mathcal{U}^\dagger(t,t_0) = \langle\alpha^{(i)}(t)|$

Plug in the facts, and get $\rho(t) = \sum_i w_i \left(\mathcal{U}(t,t_0)|\alpha^{(i)}(t_0)\rangle\right)\left(\langle\alpha^{(i)}(t_0)|\mathcal{U}^\dagger(t,t_0)\right)$

But $\mathcal{U}$ and $\mathcal{U}^\dagger$ don't depend on i, so

$\rho(t) = \mathcal{U}(t,t_0)\left(\sum_i w_i|\alpha^{(i)}(t_0)\rangle\langle\alpha^{(i)}(t_0)|\right)\mathcal{U}^\dagger(t,t_0) = \mathcal{U}(t,t_0)\,\rho(t_0)\,\mathcal{U}^\dagger(t,t_0)$ □

(b) given: $\rho(0) = |\alpha^{(i)}(0)\rangle\langle\alpha^{(i)}(0)|$. By pt. (a), $\rho(t) = \mathcal{U}(t,0)|\alpha^{(i)}(0)\rangle\langle\alpha^{(i)}(0)|\mathcal{U}^\dagger(t,0)\rangle$

$\Rightarrow \rho(t) = |\alpha^{(i)}(t)\rangle\langle\alpha^{(i)}(t)|$

Then $\rho^2(t) = |\alpha^{(i)}(t)\rangle \cancelto{1}{\langle\alpha^{(i)}(t)|\alpha^{(i)}(t)\rangle} \langle\alpha^{(i)}(t)| = |\alpha^{(i)}(t)\rangle\langle\alpha^{(i)}(t)| = \rho(t)$.

By 3.4.13, if $\rho = \rho^2$, then the ensemble is pure, so this can never evolve into a mixed state. □

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