Sakurai 3-10

(a) Start off with a generic state for a spin state:

$$|\alpha\rangle = \cos(\tfrac{\theta}{2})|+\rangle + e^{i\phi}\sin(\tfrac{\theta}{2})|-\rangle$$

where $\phi \in [-\pi, \pi]$ is the relative phase, and $\cos(\frac{\theta}{2}), \sin(\frac{\theta}{2}) \in \mathbb{R}^+$ (we can ignore overall phase).

I will write the (known) expectation values in terms of $\phi$ and $\theta$. Then I will solve for the unknowns.

$\rightarrow \langle S_x \rangle = \langle \alpha | S_x | \alpha \rangle = \frac{\hbar}{2}\left[\langle\alpha(|+\rangle\langle -| + |-\rangle\langle +|)|\alpha\rangle\right]$ by 3.21

$= \frac{\hbar}{2}\langle\alpha|+\rangle\langle -|\alpha\rangle + \frac{\hbar}{2}\langle\alpha|-\rangle\langle+|\alpha\rangle$

$= \frac{\hbar}{2}\cos(\frac{\theta}{2})e^{i\phi}\sin(\frac{\theta}{2}) + \frac{\hbar}{2}e^{-i\phi}\sin(\frac{\theta}{2})\cos(\frac{\theta}{2}) = \hbar\cos(\frac{\theta}{2})\sin(\frac{\theta}{2})\left[\frac{e^{i\phi}+e^{-i\phi}}{2}\right]$

$= \hbar\cos(\frac{\theta}{2})\sin(\frac{\theta}{2})\cos(\phi) = \frac{\hbar}{2}\sin(\theta)\cos(\phi)$ ①

$\rightarrow \langle S_y \rangle = \langle \alpha | S_y | \alpha \rangle = \frac{-i\hbar}{2}\langle\alpha|+\rangle\langle-|\alpha\rangle + \frac{i\hbar}{2}\langle\alpha|-\rangle\langle+|\alpha\rangle$ similarly

$= \frac{-i\hbar}{2}\cos(\frac{\theta}{2})e^{i\phi}\sin(\frac{\theta}{2}) + \frac{i\hbar}{2}e^{-i\phi}\sin(\frac{\theta}{2})\cos(\frac{\theta}{2})$

$= \hbar\cos(\frac{\theta}{2})\sin(\frac{\theta}{2})\sin(\phi) \Rightarrow \langle S_y \rangle = \frac{\hbar}{2}\sin(\theta)\sin(\phi)$ ②

$\rightarrow \langle S_z \rangle = \langle\alpha|S_z|\alpha\rangle = \frac{\hbar}{2}\langle\alpha|+\rangle\langle+|\alpha\rangle - \frac{\hbar}{2}\langle\alpha|-\rangle\langle-|\alpha\rangle$

$= \frac{\hbar}{2}\left(\cos^2(\frac{\theta}{2}) - \sin^2(\frac{\theta}{2})\right) = \frac{\hbar}{2}\cos(\theta)$ ③

Of course, we remember that $\langle S_x \rangle, \langle S_z \rangle$ are known, while $\langle S_y \rangle$ is not except for its sign.

$\rightarrow$ by ③, $\theta = \cos^{-1}\left(\frac{2}{\hbar}\langle S_z \rangle\right)$, where we can be sure $\theta$ is positive, since $\sin(\frac{\theta}{2}) \in \mathbb{R}^+$. Great! we know $\sin(\frac{\theta}{2})$ and $\cos(\frac{\theta}{2})$.

$\rightarrow$ Plugging that into ①, we get

$\cos\phi = \frac{2}{\hbar}\frac{\langle S_x \rangle}{\sin\theta}$ (where $\frac{2}{\hbar}\frac{\langle S_x\rangle}{\sin\theta}$ is now a known constant).

BUT!! $\phi = \pm\cos^{-1}\left(\frac{2}{\hbar}\frac{\langle S_x \rangle}{\sin\theta}\right)$ since cos is even.

How can we find the sign of $\phi$?

$\rightarrow$ equation ② to the rescue. This gives $\sin\phi = \frac{2\langle S_y \rangle}{\hbar\sin\theta}$

$\frac{2}{\hbar\sin\theta}$ is a positive constant, so the sign of $\langle S_y \rangle$ gives the sign of $\sin\phi$, and hence the sign of $\phi$.

Note the magnitude of $\phi$ can be found using just $\langle S_x \rangle$ and $\langle S_z \rangle$, but we still do need the sign of $\langle S_y \rangle$ to get the sign of $\phi$.

(b) We'll use a general matrix $\rho = \begin{bmatrix}\alpha & \beta \\ \gamma & \delta\end{bmatrix}$, and use $[S_x], [S_y], [S_z]$ to determine the unknown greek letters in our spin-1/2 density matrix using $[A] = tr(\rho A)$, and $S_x = \frac{\hbar}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $S_y = \frac{\hbar}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$, $S_z = \frac{\hbar}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

Thus, $[S_x] = tr(\rho S_x) = tr\left(\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \frac{\hbar}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) = \frac{\hbar}{2} tr\begin{bmatrix} \beta & \alpha \\ \delta & \gamma \end{bmatrix} = \frac{\hbar}{2}(\beta + \gamma)$

Similarly $[S_y] = \frac{i\hbar}{2}(\beta - \gamma)$, $[S_z] = \frac{\hbar}{2}(\alpha - \delta)$

Also, by 4.11, $tr(\rho) = 1$, so $\alpha + \delta = 1$

Thus we have enough equations to solve for $\alpha, \beta, \gamma$, and $\delta$.

Solving w/ the attached Mathematica Code:

$$\boxed{\begin{array}{ll} \alpha = \dfrac{2[S_z] + \hbar}{2\hbar}, & \beta = \dfrac{[S_x] - i[S_y]}{\hbar} \\ \gamma = \dfrac{[S_x] + i[S_y]}{\hbar}, & \delta = \dfrac{-2[S_z] + \hbar}{2\hbar} \end{array}}$$

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In[515]:= $\texttt{Solve}\left[\left\{SX == \frac{\hbar}{2}(\beta + \gamma),\ SY == \frac{i * \hbar}{2}(\beta - \gamma),\ SZ == \frac{\hbar}{2}(\alpha - \delta),\ \alpha + \delta == 1\right\}, \{\alpha, \beta, \gamma, \delta\}\right]$

Out[515]= $\left\{\left\{\alpha \to -\frac{-2\ SZ - \hbar}{2\ \hbar},\ \beta \to -\frac{-SX + i\ SY}{\hbar},\ \gamma \to -\frac{-SX - i\ SY}{\hbar},\ \delta \to -\frac{2\ SZ - \hbar}{2\ \hbar}\right\}\right\}$