Sakurai 2-8

1\. given $\langle(\Delta X)^2\rangle\langle(\Delta P)^2\rangle = \frac{\hbar^2}{4}$ and $\langle X\rangle = \langle P\rangle = 0$

thus $\langle X^2\rangle\langle P^2\rangle = \frac{\hbar^2}{4}$ because $\langle(\Delta X)^2\rangle = \langle X^2\rangle - \cancel{\langle X\rangle}^2$, for example.

$\frac{\partial}{\partial t} = \frac{1}{i\hbar}[x, H]$ is the Heisenberg equation of motion for the x operator.

and $\frac{\partial p}{\partial t} = \frac{1}{i\hbar}[p, H] = 0 \Rightarrow p = \text{constant} = p_0$.

Putting them together, $\frac{\partial x}{\partial t} = \frac{p_0}{m} \Rightarrow \int dx = \int_0^t \frac{p_0}{m} dt$

$x(t) = x_0 + \frac{p_0 t}{m}$

Then $\langle x^2(t)\rangle = \left\langle \left(x_0 + \frac{p_0 t}{m}\right)^2\right\rangle = \left(x_0^2 + \frac{p_0^2 t^2}{m^2} + \frac{x_0 p_0 t}{m} + \frac{p_0 x_0 t}{m}\right)$

$= \langle x_0^2\rangle + \frac{t^2}{m^2}\langle p_0^2\rangle + \frac{t}{m}\left[\langle x_0 p_0\rangle + \langle p_0 x_0\rangle\right]$

Since $\langle x^2\rangle\langle p^2\rangle = \frac{\hbar^2}{4} \Rightarrow \langle p_0^2\rangle = \frac{\hbar^2}{4\langle x_0^2\rangle}$

$= \langle x_0^2\rangle + \left(\frac{t\hbar}{2m}\right)^2 \frac{1}{\langle x_0^2\rangle} + \frac{t}{m}\left[\lambda^*\langle x_0 p_0\rangle + \lambda\langle p_0 x_0\rangle\right]$

where λ is completely imaginary, and we got this using $\Delta A|\alpha\rangle = \lambda\Delta B|\alpha\rangle$ from problem 1.18.

In our case we replace ΔA with A, and ΔB with B.

Then

$\langle\alpha|AB|\alpha\rangle = \langle\alpha|A\,\lambda B|\alpha\rangle = \lambda\langle\alpha|AB|\alpha\rangle$,

and since λ is completely imaginary, this just cancels off the other term $\lambda^*\langle\alpha|BA|\alpha\rangle$

Finally $\boxed{\langle x^2(t)\rangle = \langle x_0^2\rangle + \left(\frac{t\hbar}{2m}\right)^2 \frac{1}{\langle x_0^2\rangle}}$ ✓

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