Sakurai 2-6

$$\sum_{a'} |\langle a''|x|a'\rangle|^2 (E_{a'} - E_{a''}) = \sum_{a'} \langle a''|x|a'\rangle^* \langle a''|x|a'\rangle (E_{a'} - E_{a''})$$

$$= \sum_{a'} \langle a'|x^\dagger|a''\rangle \langle a''|x|a'\rangle (E_{a'} - E_{a''})$$ and since x is hermetian.

$$= \sum_{a'} \langle a'|x|a''\rangle \langle a''|x|a'\rangle (E_{a'} - E_{a''})$$

$$= \sum_{a'} \langle a'|x|a''\rangle \langle a''|x|a'\rangle E_{a'} - \sum_{a'} \langle a'|x|a''\rangle \langle a''|x|a'\rangle E_{a''}$$

$$= \sum_{a'} \langle a'|x|a''\rangle \langle a''|x E_{a'}|a'\rangle - \sum_{a'} E_{a''} \langle a'|x|a''\rangle \langle a''|x|a'\rangle$$

$$= \sum_{a'} \langle a''|xH|a'\rangle\langle a'|x|a''\rangle - E_{a''} \sum_{a'} \langle a''|x|a'\rangle \langle a'|x|a''\rangle$$

Remove completeness 1

$$= \langle a''|xHx|a''\rangle - E_{a''}\langle a''|xx|a''\rangle$$

$$= \langle a''|xHx|a''\rangle - E_{a''}\langle a''|x^2|a''\rangle \quad (\star)$$

$$= \langle a''|xHx - x^2H|a''\rangle = \langle a''|x(Hx - xH)|a''\rangle = \langle a''|x[H,x]|a''\rangle$$

or, going back to $(\star)$

$$= \langle a''|xHx - Hx^2|a''\rangle = \langle a''|(xH - Hx)x|a''\rangle = \langle a''|[x,H],x|a''\rangle$$

Thus:

$$\langle a''|x[H,x]|a''\rangle = \langle a''|[x,H],x|a''\rangle$$

which implies

$$x[H,x] = [x,H]x$$

Now take twice the original expression, and note that it equals the sum of $\langle a''|x[H,x]|a''\rangle$ and $\langle a''|[x,H],x]a''\rangle$ since we just proved this to be true:

$$2\left(\sum_{a'}|\langle a''|x|a'\rangle|^2(E_{a'}-E_{a''})\right) = \langle a''|x[H,x]|a''\rangle + \langle a''|[x,H]x|a''\rangle$$

$$= \langle a''|x[H,x]|a''\rangle - \langle a''|[H,x]x|a''\rangle$$

$$= \langle a''|x[H,x] - [H,x]x|a''\rangle = \langle a''|[x,[H,x]]|a''\rangle$$

Now we calculate the commutator:

$$[H,x] = \left(\frac{p^2}{2m} + V(x)\right)x - x\left(\frac{p^2}{2m} + V(x)\right) = \frac{p^2x}{2m} + V(x)x - x\frac{p^2}{2m} - xV(x)$$

The potential terms cancel since $[x, V(x)] = 0$

$$[H,x] = \frac{1}{2m}[p^2,x] = \frac{1}{2m}\left(p[p,x] + [p,x]p\right) = \frac{-2i\hbar p}{2m} = \frac{-i\hbar p}{m}$$

Then $[x,[H,x]] = \left[x, \frac{-i\hbar p}{m}\right] = \frac{-i\hbar}{m}[x,p] = \frac{\hbar^2}{m}$ ✓

Finally $2\sum_{a'}|\langle a''|x|a'\rangle|^2(E_{a'}-E_{a''}) = \frac{\hbar^2}{m}$

$$\Rightarrow \boxed{\sum_{a'}|\langle a''|x|a'\rangle|^2(E_{a'}-E_{a''}) = \frac{\hbar^2}{2m}}$$ ✓

10