[redacted]

4) sakurai 2-39

(a) $[\Pi_x, \Pi_y] = \left[P_x - \frac{eA_x}{c}, P_y - \frac{eA_y}{c}\right]$ as given

$= \left[P_x, P_y - \frac{eA_y}{c}\right] - \left[\frac{eA_x}{c}, P_y - \frac{eA_y}{c}\right]$ (P_x, P_y commute, as do A_x, A_y)

$= \cancel{[P_x, P_y]} - \left[P_x, \frac{eA_y}{c}\right] - \left[\frac{eA_x}{c}, P_y\right] + \cancel{\left[\frac{eA_x}{c}, \frac{eA_y}{c}\right]}$

$= -\left[P_x, \frac{e}{c}A_y\right] - \left[\frac{e}{c}A_x, P_y\right]$

$= -\frac{e}{c}\left([P_x, A_y] - [A_x, P_y]\right)$

By Gottfried, $[P_j, G(\vec{x})] = i\hbar \frac{\partial G}{\partial x_j}$, so $[P_x, A_y] = i\hbar \frac{\partial}{\partial x} A_y$ and $[P_y, A_x] = i\hbar \frac{\partial A_x}{\partial y}$

$= \frac{i\hbar e}{c}\left(\frac{\partial}{\partial x}A_y - \frac{\partial}{\partial y}A_x\right)$

Now note that $(\vec{\nabla}\times\vec{A})_i = \epsilon_{ijk}\frac{\partial}{\partial x_j}A_k$

$(\vec{\nabla}\times\vec{A})_z = \epsilon_{312}\frac{\partial}{\partial x}A_y + \underbrace{\epsilon_{321}}_{-1}\frac{\partial}{\partial y}A_x = \frac{\partial}{\partial x}A_y - \frac{\partial}{\partial y}A_x$ ✓

So $[\Pi_x, \Pi_y] = \frac{i\hbar e}{c}(\vec{\nabla}\times\vec{A})_z = \boxed{\frac{i\hbar e}{c}B_z}$ $\left(\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = (\ldots)\hat{x} - (\ldots)\hat{y} + \frac{\partial}{\partial x}A_y - \frac{\partial}{\partial y}A_x\right)$

(b) $H_u = \frac{\Pi^2}{2m} + e\phi$ and $[\Pi_x, \Pi_y] = \frac{i\hbar e}{c}\epsilon_{xyz}B_z$

and $H = \frac{P^2}{2m} + \frac{m\omega^2 x^2}{2}$ (2.3.1), $H|n\rangle = (n+\frac{1}{2})\hbar\omega|n\rangle$ for SHO's.

$H_u = \frac{\Pi_x^2 + \Pi_y^2 + \Pi_z^2}{2m}$ (no ϕ in $\vec{E}$-field-less environment)

Note that $\Pi_z = P_z - \frac{eA_z}{c} = P_z$ b/c $A_z = 0$, so $H_u = \frac{\Pi_x^2}{2m} + \frac{\Pi_y^2}{2m} + \frac{P_z^2}{2m}$

and from (a) $[\Pi_x, \Pi_y] = \frac{i\hbar e}{c}B_z \Rightarrow \left[\frac{c}{B_z e}\Pi_x, \Pi_y\right] = i\hbar$

This looks like $[x, P] = i\hbar$, so let $x \to \frac{c}{B_z e}\Pi_x$, $P \to \Pi_y$

$H = \frac{\Pi_y^2}{2m} + \frac{m\omega^2}{2}\left(\frac{c^2}{B_z^2 e^2}\Pi_x^2\right)$ Therefore $H_u = H + \frac{P_z^2}{2m}$

(so long as we let $\omega = \frac{B_z e}{mc} = \frac{|eB|}{mc}$)

Now $H_u|n\rangle = H|n\rangle + \frac{P_z^2}{2m}|n\rangle = \hbar\omega(n+\frac{1}{2})|n\rangle + \frac{\hbar^2 k^2}{2m}|n\rangle$ ← eigenvalue of the given in problem. ✓

So $E_{k,n} = \hbar\omega(n+\frac{1}{2}) + \frac{\hbar^2 k^2}{2m} = \boxed{\frac{\hbar|eB|}{mc}(n+\frac{1}{2}) + \frac{\hbar^2 k^2}{2m}}$ 10

□