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Define $Z = \int d^3x' \, K(\vec{x}', t, \vec{x}', 0)\big|_{\beta = it/\hbar} = \sum_{a'} \exp\left[\frac{-iE_{a'}t}{\hbar}\right]$ by 2.6.22.

$Z = \sum_{a'} \exp(-\beta E_{a'})$

Now we show that $\lim_{\beta\to\infty}\left(\frac{-1}{Z}\frac{\partial Z}{\partial \beta}\right) = E_0$, the ground state energy.

$$\frac{-1}{Z}\frac{\partial Z}{\partial \beta} = \frac{-1}{\sum_{a'} \exp(-\beta E_{a'})} \cdot -E_{a'}\sum_{a'} \exp(-\beta E_{a'}) = \frac{\sum_{a'} E_{a'} \exp(-\beta E_{a'})}{\sum_{a'} \exp(-\beta E_{a'})}$$

$$= \frac{E_0 e^{-\beta E_0} + E_1 e^{-\beta E_1} + E_2 e^{-\beta E_2} + \cdots}{e^{-\beta E_0} + e^{-\beta E_1} + e^{-\beta E_2} + \cdots}$$

Since $E_0 > E_1 > E_2 > \cdots$, in the limit that $\beta \to \infty$, all the terms go to 0, but the E_1, E_2, etc terms go faster, so we can ignore them. Now:

$$\lim_{\beta\to\infty}\left(\frac{-1}{Z}\frac{\partial Z}{\partial \beta}\right) \approx \lim_{\beta\to\infty}\left(\frac{E_0 \cancel{e^{-\beta E_0}}}{\cancel{e^{-\beta E_0}}}\right) = \boxed{E_0} \quad ✓ \quad \square$$

Now we demonstrate for a simple harmonic oscillator, where $E_{a'} \to E_n = (n+\frac{1}{2})\hbar\omega$

Now $$\frac{-1}{Z}\frac{\partial Z}{\partial \beta} = \frac{\hbar\omega \sum_n (n+\frac{1}{2}) e^{-\beta(n+\frac{1}{2})\hbar\omega}}{\sum_n e^{-\beta(n+\frac{1}{2})\hbar\omega}}$$

$$= \frac{\hbar\omega\left(\sum_n n e^{-\beta(n+\frac{1}{2})\hbar\omega} + \frac{1}{2}\sum_n e^{-\beta(n+\frac{1}{2})\hbar\omega}\right)}{\sum_n e^{-\beta(n+\frac{1}{2})\hbar\omega}} = \frac{\hbar\omega \sum_n n e^{-\beta n\hbar\omega}}{\sum_n e^{-\beta n\hbar\omega}} + \frac{\hbar\omega \frac{1}{2}\sum_n e^{-\beta n\hbar\omega}}{\sum_n e^{-\beta n\hbar\omega}}$$

and now take the limit (expanded again for clarity).

$$\lim_{\beta\to\infty}\left[\frac{\hbar\omega(0 + e^{-\beta\hbar\omega} + 2e^{-\beta 2\hbar\omega} + 3e^{-\beta 3\hbar\omega} + \cdots)}{(1 + e^{-\beta n\hbar\omega} + 2e^{-2\beta\hbar\omega} + 3e^{-3\beta\hbar\omega} + \cdots)} + \frac{\hbar\omega}{2}\frac{(1 + e^{-\beta\hbar\omega} + 2e^{-\beta 2\hbar\omega} + 3e^{-\beta 3\hbar\omega} + \cdots)}{(1 + e^{-\beta\hbar\omega} + 2e^{-\beta 2\hbar\omega} + 3e^{-3\beta\hbar\omega} + \cdots)}\right]$$

$$\boxed{\lim_{\beta\to\infty}\left(\frac{-1}{Z}\frac{\partial Z}{\partial \beta}\right) = \frac{\hbar\omega}{2} = E_0} \quad ✓$$

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