Sakurai 2-31

Our goal is to prove that $K(x'', t, x', t_0) = \sqrt{\dfrac{m}{2\hbar i(t-t_0)\pi}} \exp\left[\dfrac{im(x''-x')^2}{2\hbar(t-t_0)}\right]$
given that

$$K = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dp' \exp\left[\frac{ip'(x''-x')}{\hbar} - \frac{ip'^2(t-t_0)}{2m\hbar}\right]$$

$$= \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dp' \exp\left[\frac{2mip'(x''-x') - ip'^2(t-t_0)}{2m\hbar}\right]$$

Now take the exponent only, and complete the square.

$$-i(t-t_0)\left(p'^2 + \frac{-2m(x''-x')}{t-t_0}p'\right)$$

$$= -i(t-t_0)\left(p'^2 - \frac{2m(x''-x')}{t-t_0} + \frac{m^2(x''-x')^2}{(t-t_0)^2} - \frac{m^2(x''-x')}{(t-t_0)^2}\right)$$

$$= -i(t-t_0)\left[\left(p' - \frac{m(x''-x')}{t-t_0}\right)^2 - \frac{m^2(x''-x')^2}{(t-t_0)^2}\right]$$

With the square completed in the exponent, put it back in the integral.

$$= \frac{1}{2\pi\hbar}\left[\int_{-\infty}^{\infty} dp' \exp\left[\frac{-i(t-t_0)}{2m\hbar}\left(p' - \frac{m(x''-x')}{(t-t_0)^2}\right)^2\right]\right]\exp\left[\frac{im^2(x''-x')^2}{2m\hbar(t-t_0)}\right]$$

$$= \frac{1}{2\pi\hbar}\exp\left[\frac{im(x''-x')^2}{2\hbar(t-t_0)}\right]\int_{-\infty}^{\infty} dp' \exp\left[\frac{-i(t-t_0)}{2m\hbar}\left(p' - \frac{m(x''-x')}{t-t_0}\right)^2\right]$$

Now, note that $\int_{-\infty}^{\infty} dx\, e^{-a(x-b)^2} = \sqrt{\frac{\pi}{a}}$, so

$$= \frac{1}{2\pi\hbar}\frac{\sqrt{\pi}}{\sqrt{\frac{i(t-t_0)}{2m\hbar}}}\exp\left[\frac{im(x''-x')^2}{2\hbar(t-t_0)}\right] = \frac{\sqrt{2m\hbar}}{2\hbar\sqrt{\pi}\sqrt{i(t-t_0)}}\exp\left[\frac{im(x''-x')^2}{2\hbar(t-t_0)}\right]$$

$$\Rightarrow \boxed{K(x'', t, x', t_0) = \sqrt{\frac{m}{2i\hbar(t-t_0)\pi}}\exp\left[\frac{im(x''-x')^2}{2\hbar(t-t_0)}\right]} \quad \checkmark \quad \square$$

Now convert to 3-D:

By 10/2 class notes, $K(\vec{x}, t, \vec{x}', t_0') = \langle \vec{x}, t | \vec{x}', t' \rangle$

$$= \langle x, y, z, t | x', y', z', t' \rangle = (\langle x, t | (\langle y, t | (\langle z, t |)))(((| x', t' \rangle) | y', t' \rangle) | z', t' \rangle)$$

$$= \langle x, t | x', t' \rangle \langle y, t | y', t' \rangle \langle z, t | z', t' \rangle$$

just multiply "3 copies" of the answer above.

$$= \left(\frac{m}{2i\hbar(t-t_0)\pi}\right)^{3/2} \exp\left[\frac{\sim (y''-y')^2}{\sim}\right]\exp\left[\frac{\sim (x''-x')^2}{\sim}\right]\exp\left[\frac{\sim (z''-z')^2}{\sim}\right]$$

$$K(\vec{x}'', t, \vec{x}', t_0) = \left(\frac{m}{2i\hbar\pi(t-t_0)}\right)^{3/2} \exp\left[\frac{im(\vec{x}''-\vec{x}')^2}{2\hbar(t-t_0)}\right] \quad \checkmark$$

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