sakurai 2-24

given: a negative delata fn. potential $V(x) = -V_0\delta(x)$
where V_0 is real & positive

We are looking for the wave function $\langle x|a\rangle$ and Binding Energy of the ground state E_a.

Using the TISE (2.4.10)

$$\left(\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2} - V_0\delta(x)\right)\langle x|a\rangle = E_a\langle x|a\rangle$$

Solution continues on the next page.

(6)

first looking @ $x \neq 0$, $\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\langle x|a\rangle = E_a\langle x|a\rangle$

for simplicity, let $\phi^2 = \frac{-2mE_a}{\hbar^2} \Rightarrow \phi = \sqrt{\frac{-2mE_a}{\hbar^2}}$

where we will note that E_a is negative

then $\frac{\partial^2}{\partial x^2}\langle x|a\rangle + \phi^2\langle x|a\rangle = 0$

$\Rightarrow$ the solutions are of the form $Ae^{-\phi x} + Be^{\phi x} = \langle x|a\rangle$

where $Ae^{-\phi x}$ if $x>0$, and $Be^{\phi x}$ $x<0$ must be appropriate or else the wave fn. would grow w/o bound.

Also note that @ the boundary $\langle x|a\rangle_-$ and $\langle x|a\rangle_+$ must equal.

Thus $Ae^{-\phi 0} = Be^{\phi 0} \Rightarrow A = B$ which simplifies things

$$\langle x|a\rangle = \begin{cases} Ae^{-\phi x}, & x \geq 0 \\ Ae^{\phi x}, & x \leq 0 \end{cases}$$

Now deal w/ the case when $x=0$

$$\left[\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2} - V_0\,\delta(x)\right]\langle x|a\rangle = E_a\langle x|a\rangle$$

$$\Rightarrow \frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\langle x|a\rangle = (E_a + V_0\,\delta(x))\langle x|a\rangle$$

Let's integrate over an infinitesimal region around $x=0$ (from $-\varepsilon$ to ε)

$$\frac{-\hbar^2}{2m}\int_{-\varepsilon}^{\varepsilon}\frac{d^2}{dx^2}\langle x|a\rangle\,dx = \int_{-\varepsilon}^{\varepsilon}E_a\langle x|a\rangle\,dx + V_0\int_{-\varepsilon}^{\varepsilon}\delta(x)\langle x|a\rangle$$

$$\frac{-\hbar^2}{2m}\left[\left.\frac{\partial\langle x|a\rangle}{\partial x}\right|_{\varepsilon} - \left.\frac{\partial\langle x|a\rangle}{\partial x}\right|_{-\varepsilon}\right] = \int_{-\varepsilon}^{\varepsilon}E_a\langle x|a\rangle\,dx + V_0\langle 0|a\rangle$$

letting $\varepsilon \to 0$, the E_a term goes to zero.

Then, for $x>0$, $\left.\frac{\partial\langle x|a\rangle}{\partial x}\right|_{\varepsilon} = \frac{\partial}{\partial x}Ae^{-\phi x} = -\phi Ae^{-\phi x}$ and as $\varepsilon\to 0$, $= -\phi A$

for $x<0$, $\left.\frac{\partial\langle x|a\rangle}{\partial x}\right|_{\varepsilon} = \frac{\partial}{\partial x}Ae^{\phi x} = \phi Ae^{\phi x}$ and as $\varepsilon\to 0$, $= \phi A$.

Thus, plug it in, noting $\langle 0|a\rangle = Ae^{-\phi\cdot(0)} = A$

$$\frac{-\hbar^2}{2m}\left(-\phi A - \phi A\right) = V_0 A \Rightarrow V_0 = \frac{\hbar^2\phi}{m} = \frac{\hbar^2}{m}\sqrt{\frac{-2mE_a}{\hbar^2}} \Rightarrow \boxed{E_a = \frac{-mV_0^2}{2\hbar^2}}$$

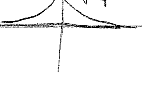
Thus the only Energy Eigenstate is the Ground State

Now, note $\langle x|a\rangle$ is normalized: $\int|\langle x|a\rangle|^2dx = 1$

The potential is even, so the wave fn. should be too. Thus $\int_0^\infty|\langle x|a\rangle|^2dx = \frac{1}{2}$

$$A^2\int_0^\infty e^{-\phi x\cdot 2}dx = 1/2 = \left.\frac{2A^2e^{-2\phi x}}{2\phi}\right|_0^\infty \Rightarrow A = \sqrt{\phi}$$

Thus $\boxed{\langle x|a\rangle = \begin{cases}\sqrt{\phi}\,e^{-\phi x} & x \geq 0 \\ \sqrt{\phi}\,e^{\phi x} & x \leq 0\end{cases}}$



(10)