Sakurai 2-19

ⓐ given $|\lambda\rangle = e^{-|\lambda|^2/2} e^{\lambda a^\dagger}|0\rangle$

First prove coheret by showing $a|\lambda\rangle = \lambda|\lambda\rangle$

$a|\lambda\rangle = \exp(-|\lambda|^2/2)\, a\, e^{\lambda a^\dagger}|0\rangle$, and since $[a, e^{\lambda a^\dagger}] = a e^{\lambda a^\dagger} - e^{\lambda a^\dagger} a$

and since $a|0\rangle = 0$, we ignore the second term.

So $a|\lambda\rangle = e^{-|\lambda|^2/2}[a, e^{\lambda a^\dagger}]|0\rangle = e^{-|\lambda|^2/2}\left[a, \sum_{n=0}^{\infty}\frac{(\lambda a^\dagger)^n}{n!}\right]|0\rangle$

$= e^{-|\lambda|^2/2}\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}[a, (a^\dagger)^n]|0\rangle = e^{-|\lambda|^2/2}\sum_{n=0}^{\infty}\frac{\lambda^n}{n!} n (a^\dagger)^{n-1}[a, a^\dagger]|0\rangle$

which is true b/c $[A, B^n] = nB^{n-1}[A,B]$

Now use 2.33 to show

$= e^{-|\lambda|^2/2}\sum_{n=0}^{\infty}\left(\frac{\lambda^n}{n!} n (a^\dagger)^{n-1}\right)|0\rangle = e^{-|\lambda|^2/2}\sum_{n=0}^{\infty}\frac{\lambda^n}{(n-1)!}(a^\dagger)^{n-1}|0\rangle$

$= e^{-|\lambda|^2/2}\lambda\sum_{n=0}^{\infty}\frac{\lambda^{n-1}}{(n-1)!}(a^\dagger)^{n-1}|0\rangle$

But $n \neq 0$, so we start @ $n=1$, or altesnatuly, since $n\to\infty$, just reindex. set $n-1 \to 0$.

$a|\lambda\rangle = e^{-|\lambda|^2/2}\lambda\left(\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}(a^\dagger)^n\right)|0\rangle = e^{-|\lambda|^2/2}\lambda e^{\lambda a^\dagger}|0\rangle$

so $a|\lambda\rangle = \lambda\left(e^{-|\lambda|^2/2}e^{\lambda a^\dagger}|0\rangle\right) = \boxed{\lambda|\lambda\rangle.\ \text{Thus Coherent}}$

Now Prove normalized: $\langle\lambda|\lambda\rangle = 1$

$\langle\lambda|\lambda\rangle = \langle 0|e^{-|\lambda|^2/2}e^{\lambda^* a}e^{-|\lambda|^2/2}e^{\lambda a^\dagger}|0\rangle$

$= Z\langle 0|\sum_{m=0}^{\infty}\frac{(\lambda^* a)^m}{m!}\sum_{n=0}^{\infty}\frac{(\lambda a^\dagger)^n}{n!}|0\rangle$ where $Z = e^{-|\lambda|^2}$

$= Z\sum_{m,n=0}^{\infty}(\lambda^*)^m(\lambda)^n\frac{1}{n!\,m!}\langle 0|a^m(a^\dagger)^n|0\rangle$

(Now, by 2.3.21 $\sqrt{n!}\,|n\rangle = (a^\dagger)^n|0\rangle$ also implies $\langle 0|a^n = \sqrt{n}\langle n|$)

$= Z\sum_{m,n=0}^{\infty}(\lambda^*)^m\lambda^n\frac{\sqrt{n!\,m!}}{n!\,m!}\langle m|n\rangle \Rightarrow m=n$

$= Z\sum_{n=0}^{\infty}(|\lambda|^2)^n\frac{1}{n!} = Ze^{|\lambda|^2} = e^{-|\lambda|^2}e^{|\lambda|^2} = \boxed{1}$ ✓

Solution continues on the next page.

ⓑ We seek $\langle(\Delta X)^2\rangle\langle(\Delta P)^2\rangle$

$\rightarrow \langle(\Delta x)^2\rangle = \langle x^2\rangle - \langle x\rangle^2$, using $x = \sqrt{\frac{\hbar}{2m\omega}}(a+a^\dagger)$

- $\langle x\rangle = \langle\lambda|x|\lambda\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left(\langle\lambda|a|\lambda\rangle + \langle\lambda|a^\dagger|\lambda\rangle\right) = \sqrt{\frac{\hbar}{2m\omega}}(\lambda+\lambda^*)$

$\Rightarrow \langle x\rangle^2 = \frac{\hbar}{2m\omega}(\lambda+\lambda^*)^2$ ✓

- $\langle x^2\rangle = \langle\lambda|xx|\lambda\rangle = \frac{\hbar}{2m\omega}\langle\lambda|(a+a^\dagger)(a+a^\dagger)|\lambda\rangle$

$= \frac{\hbar}{2m\omega}\left(\langle\lambda|aa|\lambda\rangle + 2\langle\lambda|a^\dagger a|\lambda\rangle + \langle\lambda|I|\lambda\rangle + \langle\lambda|a^\dagger a^\dagger|\lambda\rangle\right)$

(b/c $aa^\dagger + a^\dagger a = aa^\dagger - a^\dagger a + a^\dagger a + a^\dagger a = [a,a^\dagger] + 2a^\dagger a = 1 + 2a^\dagger a$)

$= \frac{\hbar}{2m\omega}\left(\lambda^2 + \lambda^{*2} + 2\lambda^*\lambda + 1\right) = \boxed{\frac{\hbar}{2m\omega}\left((\lambda^*+\lambda)^2+1\right)}$ ✓

$\rightarrow \langle(\Delta P)^2\rangle = \langle P^2\rangle - \langle P\rangle^2$

- $\langle p\rangle = \langle\lambda|P|\lambda\rangle = i\sqrt{\frac{m\omega\hbar}{2}}\langle\lambda|a^\dagger - a|\lambda\rangle$ ✓

$= -i\sqrt{\frac{m\omega\hbar}{2}}(\lambda^*-\lambda) \Rightarrow \langle p\rangle^2 = \frac{m\omega\hbar}{2}(\lambda-\lambda^*)^2$

- $\langle p^2\rangle = \langle\lambda|pp|\lambda\rangle = \frac{-m\omega\hbar}{2}\langle\lambda|(a^\dagger-a)^2|\lambda\rangle$

(now $(a^\dagger - a)^2 = a^\dagger a^\dagger + aa - a^\dagger a - aa^\dagger = a^\dagger a^\dagger + aa \cancel{- a^\dagger a} - \underbrace{aa^\dagger - a^\dagger a}_{[a^\dagger,a]=-1} + a^\dagger a$

$= a^\dagger a^\dagger + aa - 2a^\dagger a - 1$)

$= \frac{-m\omega\hbar}{2}\left(\lambda^{*2} + \lambda^2 - \lambda^*\lambda - 1\right) = \frac{m\omega\hbar}{2}\left(1-(\lambda^*-\lambda)^2\right)$ ✓

$\langle(\Delta x)^2\rangle\langle(\Delta P)^2\rangle = \frac{\hbar}{2m\omega}\left[\left((\lambda^*+\lambda)^2+1\right) - (\lambda+\lambda^*)^2\right]\cdot\frac{\hbar m\omega}{2}\left[\left(1-(\lambda^*-\lambda)^2\right) - (\lambda-\lambda^*)\right]^2$

$= \frac{\hbar}{2m\omega}\cdot\frac{m\omega\hbar}{2} = \boxed{\frac{\hbar^2}{4}}$ ✓

Minimum Uncertainty Relation.

ⓒ given $|\lambda\rangle = \sum_{n=0}^{\infty}|n\rangle\underbrace{\langle n|\lambda\rangle}_{f(n)}$ and $\langle n|\lambda\rangle = \langle n|e^{-|\lambda|^2/2}e^{\lambda a^\dagger}|0\rangle$

$\langle n|\lambda\rangle = e^{-|\lambda|^2/2}\langle n|e^{\lambda a^\dagger}|0\rangle \xrightarrow{\text{taylor expand}} e^{-|\lambda|^2/2}\langle n|\sum_{n=0}^{\infty}\frac{\lambda^n(a^\dagger)^n}{n!}|0\rangle$

$= e^{-|\lambda|^2/2}\langle n|\sum\frac{\lambda^n\sqrt{n!}}{n!}|n\rangle$

so $|\lambda\rangle = e^{-|\lambda|^2/2}\sum \lambda^n/\sqrt{n!}\,|n\rangle$

By inspection

$f(n) = e^{-|\lambda|^2/2}\lambda^n/\sqrt{n!} \Rightarrow |f(n)|^2 = e^{(|\lambda|^2/2)^2}\frac{\lambda^{2n}}{n!}$ ✓

$P = |f(n)|^2 = \exp(-|\lambda|^2)\,\lambda^{2n}/n!$

which is a Poisson PMF w the paramter of $|\lambda|^2$, and $k\rightarrow n$.

The maximum value of the distro is the only extremum, and the most probable value. We can maximize under the Log function to keep things simple.

$\ln(P) = -|\lambda|^2 + n\ln(|\lambda|^2) - \ln(n!) = 2\ln(f(n))$

Maximize: $\frac{\partial}{\partial n}\ln(P) = 0 \Rightarrow \underbrace{\frac{\partial}{\partial n}(-|\lambda|^2)}_{0} + \frac{\partial n}{\partial n}\ln(|\lambda|^2) - \frac{\partial}{\partial n}\ln(n!) = 0$

$\Rightarrow 0 = \ln|\lambda|^2 - \frac{\partial}{\partial n}\ln(n!)$

Now use Sterling's Approx ~~for~~ large (ish) n. So $\ln(n!) \approx n\ln(n) - n$

$0 = \ln(\lambda)^2 - \frac{\partial}{\partial n}(n\ln(n) - n) = \ln(\lambda)^2 - (\lambda n(n) + 1 - 1) = 0$

$\Rightarrow \ln|\lambda|^2 = \ln(n) \Rightarrow \boxed{n = |\lambda|^2}$, $\leftarrow$ most probable.

Thus the most probable energy is $\hbar\omega(n+\frac{1}{2}) = \boxed{E_{|\lambda|^2} = \hbar\omega(|\lambda|^2 + \frac{1}{2})}$

(d) Show that $a(e^{-ip\ell/\hbar}|0\rangle) = \lambda(e^{-ip\ell/\hbar}|0\rangle)$, where $\lambda = 0$. ← not necessarily zero

by 2.3.24, we rewrite p:

$a e^{\sqrt{\frac{m\hbar\omega}{2}}(-a+a^\dagger)\ell/\hbar}|0\rangle = a e^{\gamma(-a+a^\dagger)\ell}|0\rangle$ where $\gamma = \sqrt{\frac{m\omega}{2\hbar}}$

(Now $[[a,a^\dagger],a] = [1,a] = a - a = 0$
and $[[a,a^\dagger],a^\dagger] = 0$ similarly, so the condition to use the identity on the assignment sheet holds, and thus,
$e^{a+a^\dagger} = e^a e^{a^\dagger} e^{-\frac{1}{2}[a,a^\dagger]} = e^a e^{a^\dagger} e^{-\frac{1}{2}}$)

Now

$a e^{\gamma(-a+a^\dagger)\ell}|0\rangle = a(e^{-a}e^{a^\dagger}e^{-\frac{1}{2}})^{\gamma\ell}|0\rangle = e^{-\gamma\ell/2} a e^{-a\gamma\ell} e^{\sqrt{1}\gamma\ell}|1\rangle$

$= e^{-\gamma\ell/2 + \gamma\ell} a e^{-a\gamma\ell}|1\rangle = e^{-\gamma\ell/2 + \gamma\ell} a e^{-\gamma\ell}|0\rangle$

$= e^{-\gamma\ell/2} a|0\rangle = e^{-\gamma\ell/2}(0) = 0$

So we can get a coherent state by translating the ground state.

You should be led to $e^{-ip\ell/\hbar}|0\rangle = |\lambda\rangle$, or that $a[e^{-ip\ell/\hbar}|0\rangle] = \lambda[e^{-ip\ell/\hbar}|0\rangle]$ 9