Sakurai 2-13

(a) Now using the state from the last problem, let's write down the wave fn. in coordinate space: (@ $t=0$)

$\psi(x) = \langle x|\exp\left(\frac{-ipa}{\hbar}\right)|0\rangle = \int_{x'} dx' \langle x|\exp\left(\frac{-ipa}{\hbar}\right)|x'\rangle\langle x'|0\rangle$

Now, note eq. 1.6.36, where we have $T(\Delta x, x) = \exp\left(\frac{-ip\Delta x'}{\hbar}\right)$ (i.e. Generator of Translations)

when $\Delta x' \to a$, $\exp\left(\frac{-ipa}{\hbar}\right) = T(a,x)$

So, plug this into $\psi(x)$:

$\psi(x) = \int_{x'} dx' \langle x|T(a,x)|x'\rangle\langle x'|0\rangle = \int_{x'} dx' \langle x|x'+a\rangle\langle x'|0\rangle$

$= \int_{x'} dx' \,\delta((x'+a)-x)\langle x'|0\rangle = \int_{x'} dx' \,\delta(x'-(x-a))\langle x'|0\rangle = \langle x-a|0\rangle$

Using the definition given in the problem for $\langle x'|0\rangle$:

$$\boxed{\langle x-a|0\rangle = \pi^{-1/4} x_0^{-1/2} \exp\left[\frac{-1}{2}\left(\frac{x-a}{x_0}\right)^2\right] \quad w/\ x_0 \equiv \left(\frac{\hbar}{m\omega}\right)^{1/2}}$$ ✓

(b) find $P(\text{in ground state @ } t=0) = |\langle 0|\exp\left(\frac{-ipa}{\hbar}\right)|0\rangle|^2$

$= \left|\int_{x'} dx' \langle 0|\exp\left(\frac{-ipa}{\hbar}\right)|x'\rangle\langle x'|0\rangle\right|^2 = \left|\int_{x'}\int_{x''} dx'' dx' \langle 0|x''\rangle\langle x''|\exp\left(\frac{-ipa}{\hbar}\right)|x'\rangle\langle x'|0\rangle\right|^2$

and $\exp\left(\frac{-ipa}{\hbar}\right)|x'\rangle = e^{\left(\frac{-ipa}{\hbar}\right)}|x'+a\rangle$, so we bring out the exp terms and magnitude square. (they go to one).

numbers - not operators

$\langle x''|x'+a\rangle$

$= \left|\int_{x'}\int_{x''} dx'' dx' \langle 0|x''\rangle \,\delta((x'+a)-x'')\langle x'|0\rangle\right|^2$

$= \left|\int_{x'}\int_{x''} dx'' dx' \langle 0|x''\rangle \,\delta(x'-(x''-a))\langle x'|0\rangle\right|^2 = \left|\int_{x''} dx'' \langle 0|x''\rangle\langle x''-a|0\rangle\right|^2$

$= \left|\int_{x''} \pi^{-1/4} x_0^{-1/2} \exp\left(-\frac{1}{2}\left(\frac{x''}{x_0}\right)^2\right) \pi^{-1/4} x_0^{-1/2} \exp\left(\frac{-1}{2}\frac{(x''-a)^2}{x_0^2}\right) dx''\right|^2$

by the definition given in the problem. Now simplify:

$= \frac{1}{\pi x_0^2}\left|\int_{x''} \exp\left(\frac{-1}{2x_0^2}\left((x'')^2 + (x''-a)^2\right)\right)\right|^2 = \frac{1}{\pi x_0^2}\left|\int_{x''} \exp\left(\frac{-1}{2x_0^2}\left(2(x'')^2 - 2ax'' + a^2\right)\right)\right|^2$

$= \frac{1}{\pi x_0^2}\exp\left(-a^2/x_0^2\right)\left|\int_{x''} \exp\left(\frac{-1}{x_0^2}(x'')^2 + \frac{a}{x_0^2}x''\right)\right|^2 = \frac{1}{\pi x_0^2}\exp\left(-a^2/x_0^2\right)\left|\exp\left(\frac{a^2/x_0^4}{4/x_0^2}\right)\sqrt{\frac{\pi}{1/x_0^2}}\right|^2$

$= \frac{1}{\pi x_0^2} e^{-a^2/x_0^2}\left|\int_{x''} \exp\left(\frac{a^2}{4x_0^2}\right) x_0\sqrt{\pi}\right|^2 = e^{-a^2/x_0^2} e^{a^2/2x_0^2} = e^{-\frac{a^2}{x_0^2} + a^2/2x_0^2}$

$= \exp\left(\frac{-2a^2+a^2}{2x_0^2}\right) = e^{-a^2/2x_0^2} = \boxed{e^{-a^2 m\omega/2\hbar}}$ ✓

Now, we have our initial state: call it $|\psi\rangle = \exp\left(\frac{-ipa}{\hbar}\right)|0\rangle$

Now we want to find the expected value after we evolve $|\psi\rangle$ in time:

$|\langle 0|\exp\left(\frac{-iHt}{\hbar}\right)|\psi\rangle|^2$

Let's act to the left w/ the time evolution operator since $H|0\rangle = \frac{1}{2}\hbar\omega|0\rangle$ for an SHO.

Thus, $|\exp\left(\frac{-i\hbar\omega t}{2\hbar}\right)\langle 0|\psi\rangle|^2 = |\langle 0|\psi\rangle|^2 = e^{-a^2/2x_0^2}$

Just as before. $\boxed{\text{The Probability Does Not Change.}}$ ✓

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