Sakurai 2-12

Given $\exp\left(\frac{-i\hat{p}a}{\hbar}\right)|0\rangle$. Evaluate $\langle x\rangle_t$

$$\langle x\rangle = \langle 0|\exp\left(\frac{+i\hat{p}a}{\hbar}\right)x(t)\exp\left(\frac{-i\hat{p}a}{\hbar}\right)|0\rangle$$

by 2.3.45a, we have $x(t) = x(0)\cos(\omega t) + (P(0)\sin(\omega t))/m\omega$, so

$$\langle x\rangle = \langle 0|\exp\left(\frac{i\hat{p}a}{\hbar}\right)\hat{x}(0)\cos(\omega t)\exp\left(\frac{-i\hat{p}a}{\hbar}\right)|0\rangle + \langle 0|\exp\left(\frac{i\hat{p}a}{\hbar}\right)\frac{P(0)\sin(\omega t)}{m\omega}\exp\left(\frac{-i\hat{p}a}{\hbar}\right)|0\rangle$$

let's consider the first term w/o constants now:

$$\langle 0|\exp\left(\frac{i\hat{p}a}{\hbar}\right)\hat{x}\exp\left(\frac{-i\hat{p}a}{\hbar}\right)|0\rangle$$

(and since $\left[\hat{x},\exp\left(\frac{-i\hat{p}a}{\hbar}\right)\right] = a\exp\left(\frac{-i\hat{p}a}{\hbar}\right) = \hat{x}\exp\left(\frac{-i\hat{p}a}{\hbar}\right) - \exp\left(\frac{-i\hat{p}a}{\hbar}\right)\hat{x}$,

this implies $\hat{x}\exp\left(\frac{-i\hat{p}a}{\hbar}\right) = a\exp\left(\frac{-i\hat{p}a}{\hbar}\right) + \exp\left(\frac{-i\hat{p}a}{\hbar}\right)\hat{x}$)

Thus, this term becomes

$$= \langle 0|\exp\left(\frac{i\hat{p}a}{\hbar}\right)a\exp\left(\frac{-i\hat{p}a}{\hbar}\right)|0\rangle + \langle 0|\exp\left(\frac{i\hat{p}a}{\hbar}\right)\exp\left(\frac{-i\hat{p}a}{\hbar}\right)\hat{x}|0\rangle$$

$$= a\langle 0|0\rangle + \langle 0|\hat{x}|0\rangle = a + \langle x(0)\rangle$$

The second term (w/o constants) then is similarly:

$$\langle 0|\exp\left(\frac{i\hat{p}a}{\hbar}\right)\hat{P}(0)\exp\left(\frac{-i\hat{p}a}{\hbar}\right)|0\rangle = \langle 0|\cancel{\exp\left(\frac{i\hat{p}a}{\hbar}\right)\exp\left(\frac{-i\hat{p}a}{\hbar}\right)}\hat{P}(0)|0\rangle = \langle 0|\hat{p}(0)|0\rangle$$

Putting it all together:

$$\langle x(t)\rangle = \cos(\omega t)(a + \langle x(0)\rangle) + \frac{\sin(\omega t)}{m\omega}\langle 0|\hat{p}(0)|0\rangle$$

By 2.3.37, $\langle x(0)\rangle = \langle p(0)\rangle = 0$

$$\Rightarrow \boxed{\langle x(t)\rangle = \cos(\omega t)\cdot a}$$

As expected, the expected position oscillates in time.

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