Sakurai 2-10 $\boxed{}$

ⓐ given $H = \Delta(|L\rangle\langle R| + |R\rangle\langle L|)$

Let $|R\rangle \doteq \begin{bmatrix}1\\0\end{bmatrix}$ and $|L\rangle \doteq \begin{bmatrix}0\\1\end{bmatrix}$, then $H \doteq \Delta\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$

To find $|v_1\rangle$ and $|v_2\rangle$, the eigenstates, and their eigenvalues λ_1 and λ_2, just solve the usual eigenvalue problem.

$$|H - \lambda I| = 0 \Rightarrow \begin{vmatrix} -\lambda & \Delta \\ \Delta & -\lambda \end{vmatrix} = 0 \Rightarrow (\lambda^2 - \Delta^2) = 0 \Rightarrow \lambda = \pm\Delta$$

$$\lambda_1 = \Delta : \quad (H - \Delta I)v_1 = 0 \Rightarrow \begin{bmatrix} -\Delta & \Delta \\ \Delta & -\Delta \end{bmatrix} v_1 = 0 \Rightarrow \Delta\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} \doteq \Delta\begin{bmatrix} v_{12} - v_{11} \\ v_{11} - v_{12} \end{bmatrix} = 0$$

where $v_1 = \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix}$. Thus $v_{11} = v_{12}$ so $v_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$

$$\lambda_2 = -\Delta : \quad (H + \Delta I)v_2 = 0 \Rightarrow \Delta\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} v_2 = 0 \Rightarrow \Delta\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = \Delta\begin{bmatrix} v_{21} + v_{22} \\ v_{22} + v_{11} \end{bmatrix} = 0$$

Thus $-v_{21} = v_{22} \Rightarrow v_2 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$

$$\boxed{\begin{aligned} |v_1\rangle &= \tfrac{1}{\sqrt{2}}(|R\rangle + |L\rangle), \quad \lambda_1 = \Delta \\ |v_2\rangle &= \tfrac{1}{\sqrt{2}}(|R\rangle - |L\rangle), \quad \lambda_2 = -\Delta \end{aligned}}$$ ✓

ⓑ $U(t, t_0 = 0) = \exp\left[\frac{-iHt}{\hbar}\right]$

find $|\alpha, t_0 = 0, t\rangle = \exp\left(\frac{-iHt}{\hbar}\right)|\alpha\rangle = \exp\left(\frac{-iHt}{\hbar}\right)(a_1|v_1\rangle + a_2|v_2\rangle)$

where we've used $a_1|v_1\rangle + a_2|v_2\rangle$ as a generic state in a basis of eigenstates of H. (It helped me to keep track of constants a_1 and a_2 by projecting the state ket $|\alpha\rangle$ onto $|R\rangle$ and $|L\rangle$, but it's not necessary.)

$$\boxed{|\alpha, t_0 = 0, t\rangle = \frac{a_1}{\sqrt{2}}\exp\left[\frac{-i\Delta t}{\hbar}\right](|R\rangle + |L\rangle) + \frac{a_2}{\sqrt{2}}\exp\left[\frac{i\Delta t}{\hbar}\right](|R\rangle - |L\rangle)}$$ ✓

ⓒ Now, if $|\alpha\rangle = |R\rangle$, we need to find $|\langle L|R(t)\rangle|^2 = \langle L|R(t)\rangle^* \langle L|R(t)\rangle$

$= \langle R(t)|L\rangle\langle L|R(t)\rangle$

Plug in $U(t)|R\rangle = |R(t)\rangle$

$\langle R|\exp\left[\frac{iHt}{\hbar}\right]|L\rangle\langle L|\exp\left[\frac{-iHt}{\hbar}\right]|R\rangle$

Next, lets plug in completeness explicitly:

$$\left(\langle R|v_1\rangle\langle v_1|\exp\left[\tfrac{iHt}{\hbar}\right]|L\rangle + \langle R|v_2\rangle\langle v_2|\exp\left[\tfrac{iHt}{\hbar}\right]|L\rangle\right)\left(\langle L|\exp\left[\tfrac{-iHt}{\hbar}\right]|v_1\rangle\langle v_1|R\rangle + \langle L|\exp\left[\tfrac{-iHt}{\hbar}\right]|v_2\rangle\langle v_2|R\rangle\right)$$

$$= \frac{\exp\left[\frac{+i\Delta t}{\hbar}\right]}{2}\left[\left((\langle R|R\rangle + \langle R|L\rangle)(\langle R|L\rangle + \langle L|L\rangle)\right) + \left((\langle R|R\rangle - \langle R|L\rangle)(\langle R|L\rangle - \langle L|L\rangle)\right)\exp\left[\frac{-i\Delta t}{\hbar}\right]\right]$$

$$\cdot \frac{\exp\left[\frac{-i\Delta t}{\hbar}\right]}{2}\left[\left((\langle L|R\rangle + \langle L|L\rangle)(\langle R|R\rangle + \langle L|R\rangle)\right) + \left((\langle L|R\rangle - \langle L|L\rangle)(\langle R|R\rangle - \langle L|R\rangle)\right)\exp\left[\frac{i\Delta t}{\hbar}\right]\right.$$

$= \left(\frac{1}{2}\exp\left[\frac{i\Delta t}{\hbar}\right] - \frac{1}{2}\exp\left[\frac{-i\Delta t}{\hbar}\right]\right)\left(\frac{1}{2}\exp\left[\frac{-i\Delta t}{\hbar}\right] - \frac{1}{2}\exp\left[\frac{i\Delta t}{\hbar}\right]\right)$

$= \frac{1}{4} + \frac{1}{4} - \frac{1}{4}\exp\left[\frac{-2i\Delta t}{\hbar}\right] - \frac{1}{4}\exp\left[\frac{2i\Delta t}{\hbar}\right] = \boxed{\frac{1}{2} - \frac{1}{2}\cos\left(\frac{2\Delta t}{\hbar}\right)}$ ✓

(d) Here are the Schrödinger equations:

$i\hbar\frac{\partial}{\partial t}\langle R|\alpha, t_0=0, t\rangle = \langle R|H|\alpha, t_0=0, t\rangle = \Delta\langle L|\alpha(t)\rangle$

$i\hbar\frac{\partial}{\partial t}\langle L|\alpha, t_0=0, t\rangle = \langle L|H|\alpha, t_0=0, t\rangle = \Delta\langle R|\alpha(t)\rangle$

And differentiate again to get, for the $|R\rangle$ equation:

$-\frac{\hbar^2}{\Delta^2}\frac{\partial^2}{\partial t^2}\langle R|\alpha(t)\rangle = \langle R|\alpha(t)\rangle$ and the same for $|L\rangle$

which has solutions $\boxed{a_1 \sin\left(\frac{\Delta t}{\hbar}\right) + a_2\cos\left(\frac{\Delta t}{\hbar}\right) = \langle R|\alpha(t)\rangle = \langle L|\alpha(t)\rangle}$

Thus, for our arbitrary state $|\alpha\rangle$

$|\alpha\rangle = \frac{a_1}{\sqrt{2}}\left(\cos\left(\frac{-\Delta t}{\hbar}\right) + i\sin\left(\frac{-\Delta t}{\hbar}\right)\right)\underbrace{(|R\rangle + |L\rangle)}_{|V_1\rangle} + \frac{a_2}{\sqrt{2}}\left(\cos\left(\frac{\Delta t}{\hbar}\right) + i\sin\left(\frac{\Delta t}{\hbar}\right)\right)\underbrace{(|R\rangle - |L\rangle)}_{|V_2\rangle}$

$= \frac{2}{\sqrt{2}}\left[\cos\left(\frac{\Delta t}{\hbar}\right)|R\rangle - i\sin\left(\frac{\Delta t}{\hbar}\right)|L\rangle\right]$ ✓

which is what you'd expect from b.

(e) $\langle\alpha(t)|\alpha(t)\rangle = 1$ (or @ least it should!)

$\langle\alpha(t)|\alpha(t)\rangle = \langle\alpha(0)|U^\dagger U|\alpha(0)\rangle = \langle\alpha_0|\exp\left[\frac{iH^\dagger t}{\hbar}\right]\exp\left[\frac{-iHt}{\hbar}\right]|\alpha_0\rangle$

and $H^\dagger = \Delta|R\rangle\langle L| \neq H$ (i.e. it's not Hermitian) ✓

thus $\exp\left[\frac{iH^\dagger t}{\hbar}\right]\exp\left[\frac{-iHt}{\hbar}\right] \neq 1$, so $\langle\alpha_0|U^\dagger U|\alpha_0\rangle \neq 1$

and probability isn't conserved.

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