70/70

sakuraai 1
2-1

given $H = -\left(\frac{eB}{mc}\right) S_z = \omega S_z$

First write the equations of motion:

$$\frac{d S_x(t)}{dt} = \frac{1}{i\hbar}\left[S_x(t), \omega S_z\right] = \frac{\omega}{i\hbar}\left[S_x, S_z\right] = \frac{\omega}{i\hbar}\left(-i\hbar S_y\right) = -\omega S_y(t)$$

$$\frac{d S_y(t)}{dt} = \frac{1}{i\hbar}\left[S_y(t), \omega S_z\right] = \frac{\omega}{i\hbar}\left[S_y, S_z\right] = \frac{\omega}{i\hbar}\left(i\hbar S_x\right) = \omega S_x(t)$$

$$\frac{d S_z(t)}{dt} = \frac{1}{i\hbar}\left[S_z(t), \omega S_z\right] = 0$$

The best way to solve this coupled system of ODEs (namely, the S_x and S_y equations) is to differentiate again...

$$\frac{d^2 S_x(t)}{dt^2} = \frac{1}{i\hbar}\left[-\omega S_y(t), \omega S_z(t)\right] = \frac{-\omega^2}{i\hbar}\left(i\hbar S_x(t)\right) = \omega^2 S_x(t)$$

Now we have $S_x(t)'' - \omega^2 S_x(t) = 0$ which has solutions $S_x(t) = C_1 \sin(t\sqrt{\omega^2}) + C_2 \cos(t\sqrt{\omega^2})$

$\Rightarrow S_x(t) = C_1 \sin(\omega t) + C_2 \cos(\omega t)$

Now use the initial condition $S_x(t=0) = S_{x0}$ which implies $C_2 = S_{x0}$

Now use the condition found above: $\frac{d S_x(t)}{dt} = -\omega S_y(t)$

$\frac{S_x(t)}{dt} = \omega C_1 \cos(\omega t) - \omega S_{x0} \sin(\omega t) = -\omega S_y(t)$

And finally, $S_y(t=0) = S_{y0} \Rightarrow \omega C_1 - 0 = -\omega S_{y0} \Rightarrow C_1 = -S_{y0}$

$\boxed{S_x(t) = -S_{y0} \sin(\omega t) + S_{x0} \cos(\omega t)}$ ✓

Now for $S_y(t)$, proceed as before:

$\frac{d^2 S_y(t)}{dt^2} = -\omega^2 S_y(t) \Rightarrow S_y(t) = C_1 \sin(\omega t) + C_2 \cos(\omega t)$

use $S_y(t=0) = S_{y0} \Rightarrow C_2 = S_{y0}$

and $\frac{d S_y(t)}{dt} = \omega S_x \Rightarrow \omega S_x(t) = C_1 \omega \cos(\omega t) - C_2 \sin(\omega t)$

Then finally $S_x(t=0) = S_x \Rightarrow S_{0x} = C_1$

$\boxed{S_y(t) = S_{0x} \sin(\omega t) + S_{0y} \cos(\omega t)}$ ✓

For the $S_z(t)$ case, clearly, the derivative of a constant is zero, and $S_z(t=0) = S_{z0}$. Thus $\boxed{S_z(t) = S_{z0}}$ ✓

10