Sakurai 9

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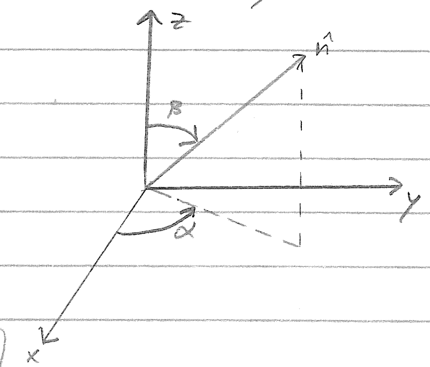
The goal: construct $|S_{\hat{n}}+\rangle$ such that $S_n|S_n+\rangle = \frac{\hbar}{2}|S_n+\rangle$

Basically, find the spin up state in an arbitrary $\hat{n}$-direction using the S_z basis.



$$S_n = S\cdot\hat{n} = (S_x + S_y + S_z)\cdot(n_x + n_y + n_z)$$
$$= \Big[\tfrac{\hbar}{2}(|+\rangle\langle -| + |-\rangle\langle +|)$$
$$+\tfrac{\hbar}{2}(-i|+\rangle\langle -| + i|-\rangle\langle +|)$$
$$+\tfrac{\hbar}{2}(|+\rangle\langle +| - |-\rangle\langle -|)\Big]$$
$$\cdot\left[\cos\alpha\sin\beta\,\hat{x} + \sin\alpha\sin\beta\,\hat{y} + \cos\beta\,\hat{z}\right]$$

$$S_n = \tfrac{\hbar}{2}\cos\alpha\sin\beta(|+\rangle\langle -| + |-\rangle\langle +|) - \tfrac{\hbar i}{2}\sin\alpha\sin\beta(|+\rangle\langle -| - |-\rangle\langle +|)$$
$$+ \tfrac{\hbar}{2}\cos\beta(|+\rangle\langle +| - |-\rangle\langle -|)$$

We want $|S_n+\rangle$ in the S_z-basis, so we must be able to write $|S_n+\rangle$ as a linear combination of the $|+\rangle$ and $|-\rangle$ states.

let $|S_n+\rangle = a|+\rangle + b|-\rangle$

So...

$$S_{\hat{n}}|S_{\hat{n}}+\rangle = \tfrac{\hbar}{2}\Big[\cos\alpha\sin\beta\, a(|+\rangle\langle -|+\rangle + |-\rangle\langle +|+\rangle)$$
$$-i\sin\alpha\sin\beta\, a(|+\rangle\langle -|-\rangle + |-\rangle\langle +|-\rangle) + \cos\beta\, a(|+\rangle\langle +|+\rangle - |-\rangle\langle -|+\rangle)$$
$$+\cos\alpha\sin\beta\, b(|+\rangle\langle -|-\rangle + |-\rangle\langle +|-\rangle)$$
$$+i\sin\alpha\sin\beta\, b(|+\rangle\langle -|-\rangle - |-\rangle\langle +|-\rangle) + \cos\beta\, b(|+\rangle\langle +|-\rangle - |-\rangle\langle -|-\rangle)\Big]$$

$$= \tfrac{\hbar}{2}\Big[[a(\cos\alpha\sin\beta + i\sin\alpha\sin\beta) - b\cos\beta]|-\rangle$$
$$+[b(\cos\alpha\sin\beta - i\sin\alpha\sin\beta) + a\cos\beta]|+\rangle\Big]$$

$$= \tfrac{\hbar}{2}(b|-\rangle + a|+\rangle)$$

equating the coefficients, we get

$$a = b\sin\beta e^{-i\alpha} + a\cos\beta \qquad \Rightarrow \qquad b = \frac{a\sin\beta e^{i\alpha}}{1+\cos\beta}$$
$$b = a\sin\beta e^{i\alpha} - b\cos\beta$$

We impose the normalization condition on the coefficients $|a|^2 + |b|^2 = 1$

$$\Rightarrow\quad a^2 + \frac{a^2\sin^2\beta e^{i\alpha}e^{-i\alpha}}{1+\cos\beta} = 1 \quad\Rightarrow\quad 1 = a^2\left(1 + \frac{\sin^2\beta}{(1+\cos\beta)^2}\right)$$

$$a^2 = 1 \Big/ \left(1 + \frac{\sin^2\beta}{(1+\cos\beta)^2}\right) = \frac{(1+\cos\beta)^2}{1+2\cos\beta+\underbrace{\cos^2\beta+\sin^2\beta}_{1}}$$

$$= \frac{(1+\cos\beta)^2}{2+2\cos\beta} = \frac{1+\cos\beta}{2} = \cos^2\left(\frac{\beta}{2}\right)$$ by the half angle formula.

$$\Rightarrow a = \cos\left(\frac{\beta}{2}\right)$$

Plug that into equation ★ (previous page) to get b.

$$b = \frac{a\sin\beta\, e^{i\alpha}}{1+\cos\beta} = \frac{\cos\left(\frac{\beta}{2}\right)\sin\beta\, e^{i\alpha}}{1+\cos\beta} = \frac{\cos\left(\frac{\beta}{2}\right)\sin\beta\, e^{i\alpha}}{2\cos^2\left(\frac{\beta}{2}\right)}$$ (half angle)

$$= \frac{\sin\beta\, e^{i\alpha}}{2\cos\left(\frac{\beta}{2}\right)}$$

Using a double angle formula, $\sin(\beta) = 2\sin\left(\frac{\beta}{2}\right)\cos\left(\frac{\beta}{2}\right)$

$$= \frac{\cancel{2}\sin\left(\frac{\beta}{2}\right)\cancel{\cos\left(\frac{\beta}{2}\right)}\, e^{i\alpha}}{\cancel{2\cos\left(\frac{\beta}{2}\right)}} = \sin\left(\frac{\beta}{2}\right) e^{i\alpha}.$$

✓ Thus, $\boxed{|S_n^+\rangle = \cos\left(\frac{\beta}{2}\right)|+\rangle + \sin\left(\frac{\beta}{2}\right)e^{i\alpha}|-\rangle}$