Sakurai 1-4

10/10

ⓐ Prove: $tr(XY) = tr(YX)$ where X, Y are operators.

$tr(XY) = \sum_{a'} \langle a'|XY|a'\rangle$ by definition of the trace

$= \sum_{a'} \langle a'|X\left(\sum_{b'} |b'\rangle\langle b'|\right)Y|a'\rangle$ by the completeness theorem

$= \sum_{a'b'} \langle a'|X|b'\rangle\langle b'|Y|a'\rangle = \sum_{a'b'} \langle b'|Y|a'\rangle\langle a'|X|b'\rangle$ ✓

$= \sum_{b'} \langle b'|Y\underbrace{\left(\sum_{a'}|a'\rangle\langle a'|\right)}_{1}X|b'\rangle = \sum_{b'} \langle b'|YX|b'\rangle = tr(YX)$ □

ⓑ Prove: $(XY)^\dagger = Y^\dagger X^\dagger$

by 1.2.24, $X|\alpha\rangle \xleftrightarrow{DC} \langle\alpha|X^\dagger$. Thus $XY|\alpha\rangle \leftrightarrow \langle\alpha|(XY)^\dagger$

we also have $XY|\alpha\rangle = X(Y|\alpha\rangle) \longleftrightarrow (\langle\alpha|Y^\dagger)X^\dagger = \langle\alpha|Y^\dagger X^\dagger$

Therefore $\langle\alpha|(XY)^\dagger = \langle\alpha|Y^\dagger X^\dagger$, and $(XY)^\dagger = Y^\dagger X^\dagger$ ✓

ⓒ $\exp[if(A)] \approx 1 + if(A) + \cdots + \frac{i^n f^n(A)}{n!}$

insert completeness twice: $= \sum_j\left(\sum_i |a_i\rangle\langle a_i|\left(1 + if(A) + \cdots + \frac{i^n f^n(A)}{n!}\right)|a_j\rangle\langle a_j|\right)$

we can distribute b/c of linearity:

$= \sum_{ij} |a_i\rangle\left[\langle a_i|a_j\rangle + \langle a_i|if(A)|a_j\rangle + \cdots + \langle a_i|\frac{i^n f^n(A)}{n!}|a_j\rangle\right]\langle a_j|$

now, $f(A) = f(a)|a\rangle$ b/c a is an eigenvalue of A.

$\sum_{ij} |a_i\rangle\left(\langle a_i|a_j\rangle + if(a_j)\langle a_i|a_j\rangle + \cdots + \frac{i^n f^n(a_j)}{n!}\langle a_i|a_j\rangle\right)\langle a_j|$

$= 0$ if $i \neq j$. otherwise $= \boxed{\sum_i |a_i\rangle\langle a_i| \exp[if(a_i)]}$ ✓