Sakurai 1.30

ⓐ We know from #29 (on the last hmwk), $[x_i, G(\vec{p})] = i\hbar \frac{\partial G}{\partial p_i}$

Thus, $[x_i, T(\vec{l})] = i\hbar \frac{\partial}{\partial p_i} \exp\left[\frac{-i\vec{p}\cdot\vec{l}}{\hbar}\right]$

now we actually carry out the dot prod. in the exponent, and thus we get a bunch of multiplied exponentials:

$$= i\hbar \frac{\partial}{\partial p_i} \prod_j \exp\left[\frac{-ip_j l_j}{\hbar}\right] = i\hbar\left(\frac{-il_i}{\hbar}\right)\exp\left[\frac{-ip_i l_i}{\hbar}\right]$$

where the p_ith derivative has killed off all but the i^{th} entry.

$$= l_i \exp\left[\frac{-ip_i l_i}{\hbar}\right] = \boxed{l_i T(\vec{l})}$$ ✓

ⓑ $\langle\psi|x_i|\psi\rangle = \langle x_i\rangle$. Let's see how this $\overset{\text{1-D}}{\vee}$ quantity changes when we translate $|\psi\rangle$. We know $T(l_i)|\psi\rangle = |\psi + l_i\rangle$

Thus $\langle\psi|T^\dagger(l_i)x_iT(l_i)|\psi\rangle = \langle\psi+l_i|x_i|\psi+l_i\rangle$

But $\langle\psi|T^\dagger(l_i)x_iT(l_i)|\psi\rangle = \langle\psi|T^\dagger(l_i)T(l_i)x_i|\psi\rangle + \langle\psi|T^\dagger(l_i)[x_i, T(l_i)]|\psi\rangle$

as well b/c $AB = BA + [A,B]$.

$\therefore \langle\psi|T^\dagger(l_i)x_iT(l_i)|\psi\rangle = \langle\psi|x_i|\psi\rangle + \langle\psi|T^\dagger(l_i)l_iT(l_i)|\psi\rangle$

by the fact that $T^\dagger(l_i)T(l_i) = 1$, and by part ⓐ.

and now

$\langle\psi|T^\dagger(l_i)x_iT(l_i)|\psi\rangle = \langle x_i\rangle + l_i\cancelto{1}{\langle\psi|\psi\rangle}$

$\langle\psi|T^\dagger(l_i)x_iT(l_i)|\psi\rangle = \langle x_i\rangle + l_i$

Putting it all together, $\langle\psi+l_i|x_i|\psi+l_i\rangle = \langle x_i\rangle + l_i$

This must hold for all the components i, j, k, etc., so

$$\boxed{\langle\psi+l|x|\psi+l\rangle = \langle x\rangle + l}$$ ✓

$\frac{10}{10}$