⑥

Sobusai 1-24

(a) Per 3.2.28, $\chi = \begin{bmatrix} c_+ \\ c_- \end{bmatrix}$, then $\frac{1}{\sqrt{2}}(I + i\sigma_x)\chi = \frac{1}{\sqrt{2}}\begin{bmatrix} c_+ \\ c_- \end{bmatrix} + \frac{i}{\sqrt{2}}\begin{bmatrix} c_- \\ c_+ \end{bmatrix}$

if we simply carry out the matrix multiplication.

Therefore $\frac{1}{\sqrt{2}}(I + i\sigma_x)\chi = \frac{1}{\sqrt{2}}\left(\begin{bmatrix} c_+ \\ c_- \end{bmatrix} + i\begin{bmatrix} c_- \\ c_+ \end{bmatrix}\right)$ remember this.

where χ is the 2-component spinor.

Now, by 3.2.45, the 2x2 rotation matrix is

$$\begin{bmatrix} \cos\left(\frac{\phi}{2}\right) - in_z \sin\left(\frac{\phi}{2}\right) & (-in_x - n_y)\sin\left(\frac{\phi}{2}\right) \\ (-in_x + n_y)\sin\left(\frac{\phi}{2}\right) & \cos\left(\frac{\phi}{2}\right) + in_z \sin\left(\frac{\phi}{2}\right) \end{bmatrix} \overbrace{\begin{bmatrix} c_+ \\ c_- \end{bmatrix}}^{\chi}$$

Multiplying again by the 2-component spinor, and setting $\phi \to -\pi/2$ and $n_z = n_y = 0$, $n_x \to 1$ as set out in the problem, we get

$$\begin{bmatrix} c_+ \cos(-\pi/4) - ic_- \sin(\pi/4) \\ ic_+ \sin(-\pi/4) + c_- \cos(-\pi/4) \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} c_+ + ic_- \\ ic_+ + c_- \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix} c_+ + ic_- \\ c_- + ic_+ \end{bmatrix} = \frac{1}{\sqrt{2}}\left(\begin{bmatrix} c_+ \\ c_- \end{bmatrix} + i\begin{bmatrix} c_- \\ c_+ \end{bmatrix}\right) = \frac{1}{\sqrt{2}}(I + i\sigma_x)\chi \checkmark$$

Thus:

$$\frac{1}{\sqrt{2}}(I + i\sigma_x) \doteq \begin{bmatrix} \cos\left(\frac{\phi}{2}\right) - in_z \sin\left(\frac{\phi}{2}\right) & (-in_x - n_y)\sin\left(\frac{\phi}{2}\right) \\ (-in_x + n_y)\sin\left(\frac{\phi}{2}\right) & \cos\left(\frac{\phi}{2}\right) + in_z \sin\left(\frac{\phi}{2}\right) \end{bmatrix} \quad \square$$

(b) $\langle S_y^\pm | S_z | S_y^\pm \rangle$ are the matrix elements of S_z in the $|S_y^\pm\rangle$ basis.

where $S_z = \frac{\hbar}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $|S_y \pm\rangle = \frac{1}{\sqrt{2}}(|+\rangle \pm i|-\rangle)$ by 1.4.17b.

$$\langle S_y^\pm | S_z | S_y^\pm \rangle = \begin{bmatrix} \langle S_y^+ | S_z | S_y^+ \rangle & \langle S_y^+ | S_z | S_y^- \rangle \\ \langle S_y^- | S_z | S_y^+ \rangle & \langle S_y^- | S_z | S_y^- \rangle \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix} (\langle +| - i\langle -|)\, S_z\, (|+\rangle + i|-\rangle) & (\langle +| - i\langle -|)\, S_z\, (|+\rangle - i|-\rangle) \\ (\langle +| + i\langle -|)\, S_z\, (|+\rangle + i|-\rangle) & (\langle +| + i\langle -|)\, S_z\, (|+\rangle - i|-\rangle) \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix} \frac{\hbar}{2}(\langle +| - i\langle -|)(|+\rangle - i|-\rangle) & \frac{\hbar}{2}(\langle +| - i\langle -|)(|+\rangle + i|-\rangle) \\ \frac{\hbar}{2}(\langle +| + i\langle -|)(|+\rangle - i|-\rangle) & \frac{\hbar}{2}(\langle +| + i\langle -|)(|+\rangle + i|-\rangle) \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix} \frac{\hbar}{2}(1-1) & \frac{\hbar}{2}(1+1) \\ \frac{\hbar}{2}(1+1) & \frac{\hbar}{2}(1-1) \end{bmatrix} = \boxed{\frac{\hbar}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}} \checkmark$$

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