Sakurai 1-23

Given $A \doteq \begin{bmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -a \end{bmatrix}$ $B \doteq \begin{bmatrix} b & 0 & 0 \\ 0 & 0 & -ib \\ 0 & ib & 0 \end{bmatrix}$ $a, b \in \mathbb{R}$

ⓐ Does B exhibit a degenerate spectrum?

→ Find the eigenvalues: $B - \lambda I = \begin{bmatrix} b-\lambda & 0 & 0 \\ 0 & -\lambda & -ib \\ 0 & ib & -\lambda \end{bmatrix}$

$$\det(B-\lambda I) = (b-\lambda)(\lambda^2 - b^2)$$

eigenvalues: $\lambda = b$, $\lambda = b$, $\lambda = -b$

$\boxed{\text{Two-Fold Degenerate.}}$ Thus, the answer is Yes.

Solution continues on the next page

(b) Show A and B commute:

$$[A,B] = AB - BA = \begin{bmatrix} ab & 0 & 0 \\ 0 & 0 & iab \\ 0 & -iab & 0 \end{bmatrix} - \begin{bmatrix} ab & 0 & 0 \\ 0 & 0 & iab \\ 0 & -iab & 0 \end{bmatrix} = \begin{bmatrix} ab-ab & 0-0 & 0-0 \\ 0-0 & 0-0 & iab-iab \\ 0-0 & -iab+iab & 0-0 \end{bmatrix}$$

$$\Rightarrow [A,B] \doteq \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \boxed{[A,B] = 0}$$ ✓

(c) Find a new set of eigenkets, simultaneous w/ the $\{|a\rangle\}$ eigenkets.

$B - \lambda I = 0$ can be represented as

$$\left[\begin{array}{ccc|c} b-\lambda & 0 & 0 & 0 \\ 0 & -\lambda & -ib & 0 \\ 0 & ib & -\lambda & 0 \end{array}\right]$$

find $\vec{v}_1, \vec{v}_2, \vec{v}_3$. The eigenvectors.

for $\lambda = b$

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & -b & -ib & 0 \\ 0 & ib & -b & 0 \end{array}\right]$$

find $\vec{v}_1 = \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix}$ line one says: $0\eta_1 + 0\eta_2 + 0\eta_3 = 0$

so we choose $\eta_1 = 1$, $\eta_2, \eta_3 = 0$

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$ ✓

OR

let $\eta_1 = 0$, then η_2, η_3 are free so far.

lines 2 and 3 say: $\eta_2(-b) + \eta_3(-ib) = 0 \Rightarrow \eta_2 = \eta_3 \frac{ib}{-b} \Rightarrow \eta_2 = \eta_3(-i)$

$$\vec{V}_2 = \begin{bmatrix} 0 \\ \eta_3(-i) \\ \eta_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -i \\ 1 \end{bmatrix}$$ Normalize: $\vec{V}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ -i \\ 1 \end{bmatrix}$ ✓

for $\lambda = -b$

$$\left[\begin{array}{ccc|c} 2b & 0 & 0 & 0 \\ 0 & b & -ib & 0 \\ 0 & ib & b & 0 \end{array}\right]$$

Now $2b\eta_1 = 0 \Rightarrow \eta_1 = 0$

$\eta_2 b = \eta_3 ib \Rightarrow \eta_2 = \eta_3 i$

$\eta_2 ib = -\eta_3 b \Rightarrow \eta_2 = \eta_3 i$

$$\rightarrow \vec{V}_3 = \begin{bmatrix} 0 \\ \eta_3 i \\ \eta_3 \end{bmatrix} \rightarrow \underbrace{\frac{1}{\sqrt{2}}}_{\text{Normalize}} \begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix}$$ ✓

Find the eigenvalues of $\vec{v}_1, \vec{v}_2, \vec{v}_3$ w/ respect to A (first) then B.

$$A\vec{v}_1 = \begin{bmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -a \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$ $\rightarrow$ eigenvalue a. ✓

$$A\vec{V}_2 = \begin{bmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -a \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ -i \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ ai \\ -a \end{bmatrix} = \frac{-a}{\sqrt{2}}\begin{bmatrix} 0 \\ -i \\ 1 \end{bmatrix}$$ $\rightarrow$ eigenvalue $-a$ ✓

$$A\vec{v}_3 = \begin{bmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -a \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -ia \\ -a \end{bmatrix} = \frac{-a}{\sqrt{2}} \begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix} \rightarrow \text{eigenvalue } -a$$

Now for B:

$$B\vec{v}_1 = \begin{bmatrix} b & 0 & 0 \\ 0 & 0 & -ib \\ 0 & ib & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} b \\ 0 \\ 0 \end{bmatrix} = b\vec{v}_1 \rightarrow \text{eigenvalue } b$$

$$B\vec{v}_2 = \begin{bmatrix} b & 0 & 0 \\ 0 & 0 & -ib \\ 0 & ib & 0 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ -i \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -ib \\ b \end{bmatrix} = \frac{b}{\sqrt{2}} \begin{bmatrix} 0 \\ -i \\ 1 \end{bmatrix} = b\vec{v}_2 \rightarrow \text{eigenvalue } b.$$

$$B\vec{v}_3 = \begin{bmatrix} b & 0 & 0 \\ 0 & 0 & -ib \\ 0 & ib & 0 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -ib \\ -b \end{bmatrix} = \frac{-b}{\sqrt{2}} \begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix} = -b\vec{v}_3 \rightarrow \text{eigenvalue } -b$$

eigenvector	eigenvalue w.r.t. A	eigenvalue w.r.t. B
$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$	a	b
$\vec{v}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ -i \\ 1 \end{bmatrix}$	$-a$	b
$\vec{v}_3 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix}$	$-a$	$-b$

Does this specification of eigenvalues completely characterize each eigenstate?

$\boxed{\text{Yes!}}$ We have 3 distinct eigenvalues corresponding to each of the 3 distinct eigenstates shared by A and B.

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Any one set (e.g. $b, b, -b$) do not totally specify the system b/c we don't know which state b corresponds to. However the degeneracies do not line up. This is good. It means that between the two eigenvalues, we can always be sure of the state we are in.

Since you have distinct pairs of eigenvalues, you're golden!