Sakurai 1-2

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$X = a_0 + \vec{\sigma}\cdot\vec{a}$ is given, where $\sigma = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \end{bmatrix}$ and $\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$

Thus, $X = a_0 + a_1\sigma_x + a_2\sigma_y + a_3\sigma_z = \begin{bmatrix} a_0 & 0 \\ 0 & a_0 \end{bmatrix} + \begin{bmatrix} 0 & a_1 \\ a_1 & 0 \end{bmatrix} + i\begin{bmatrix} 0 & -a_2 \\ a_2 & 0 \end{bmatrix} + \begin{bmatrix} a_3 & 0 \\ 0 & a_3 \end{bmatrix}$

(a) To find the relationship between $tr(X)$, and $tr(\sigma_k X)$; and a_0, a_1, a_2, and a_3, we just have to find the traces.

$X = \begin{bmatrix} a_0 + a_3 & a_1 - ia_2 \\ a_1 + ia_2 & a_0 - a_3 \end{bmatrix} \Rightarrow tr(X) = 2a_0.$

$\sigma_1 X = \begin{bmatrix} a_1 + ia_2 & a_0 + a_3 \\ a_0 + a_3 & a_1 - ia_2 \end{bmatrix} \Rightarrow tr(\sigma_1 X) = 2a_1$

4 continued

$\sigma_2 X = \begin{bmatrix} -i(a_1 + ia_2) & \sim \\ \sim & i(a_1 - ia_2) \end{bmatrix} \Rightarrow tr(\sigma_2 X) = 2a_2$

$\sigma_3 X = \begin{bmatrix} a_0 + a_3 & \sim \\ \sim & -(a_0 - a_3) \end{bmatrix} \Rightarrow tr(\sigma_3 X) = 2a_3$

Thus $\boxed{a_0 = \frac{1}{2} tr(X), \text{ and } a_k = \frac{1}{2} tr(\sigma_k X)}$

(b) recall $X = \begin{bmatrix} a_0 + a_3 & a_1 - ia_2 \\ a_1 + ia_2 & a_0 - a_3 \end{bmatrix}$. Our task is to write a_0 and a_k in terms of the matrix elements X_{ij}.

$X_{11} = a_0 + a_3 \Rightarrow a_0 = X_{11} - a_3 \Rightarrow a_3 = X_{11} - a_0$

$X_{12} = a_1 - ia_2 \Rightarrow a_1 = X_{12} + ia_2 \Rightarrow ia_2 = -X_{12} + a_1$

$X_{21} = a_1 + ia_2 \Rightarrow a_1 = X_{21} - ia_2 \Rightarrow ia_2 = X_{21} - a_1$

$X_{22} = a_0 - a_3 \Rightarrow a_0 = X_{22} + a_3 \Rightarrow a_3 = -X_{22} + a_0$

so $X_{11} - X_{22} = 2a_3 \Rightarrow a_3 = \frac{1}{2}(X_{11} - X_{22})$

$X_{11} - a_0 = -X_{22} + a_0 \Rightarrow a_0 = \frac{1}{2}(X_{11} + X_{22})$

$X_{12} + ia_2 = X_{21} - ia_2 \Rightarrow a_2 = \frac{1}{2i}(X_{21} - X_{12})$

$-X_{12} + a_1 = X_{21} - a_1 \Rightarrow a_1 = \frac{1}{2}(X_{21} + X_{12})$