Sakurai 1-19

ⓐ Task #1: compute $\langle(\Delta S_x)^2\rangle \equiv \langle S_x^2\rangle - \langle S_x\rangle^2$ w/ respect to $|S_z^+\rangle$ state.

$$\to S_x^2 = \left(\frac{\hbar}{2}[|+\rangle\langle-| + |-\rangle\langle+|]\right)^2 = \frac{\hbar^2}{4}\Big[(|+\rangle\langle-|)(|+\rangle\langle-|) + (|-\rangle\langle+|)(|-\rangle\langle+|) + (|+\rangle\langle-|)(|-\rangle\langle+|) + (|-\rangle\langle+|)(|+\rangle\langle-|)\Big]$$

$$S_x^2 = \frac{\hbar^2}{4}\left[|+\rangle\langle+| + |-\rangle\langle-|\right]$$

$$\langle S_x^2\rangle = \langle+|S_x^2|+\rangle = \frac{\hbar^2}{4}\left[\langle+|+\rangle\langle+|+\rangle + \cancel{\langle+|-\rangle\langle-|+\rangle}\right] = \frac{\hbar^2}{4}$$

$$\langle S_x\rangle^2 = \langle+|S_x|+\rangle^2 = \left(\frac{\hbar^2}{4}\left[\langle+|+\rangle\cancel{\langle-|+\rangle} + \cancel{\langle+|-\rangle}\langle+|+\rangle\right]\right)^2 = 0$$

Thus $\langle(\Delta S_x)^2\rangle = \langle S_x^2\rangle - \langle S_x\rangle^2 = \frac{\hbar^2}{4} - 0 = \boxed{\frac{\hbar^2}{4}}$ ✓

Task #2: check $\langle(\Delta S_x)^2\rangle\langle(\Delta S_y)^2\rangle \geq \frac{1}{4}|\langle[S_x, S_y]\rangle|^2$

we have $\langle(\Delta S_x)^2\rangle = \frac{\hbar^2}{4}$. We need $\langle(\Delta S_y)^2\rangle \equiv \langle S_y^2\rangle - \langle S_y\rangle^2$

- $\langle S_y^2\rangle = \left\langle\left(\frac{\hbar}{2}[-i(|+\rangle\langle-|) + i(|-\rangle\langle+|)]\right)^2\right\rangle$

$$= \left\langle\frac{\hbar^2}{4}\left[-|+\rangle\langle-|+\rangle\langle-| - |-\rangle\langle+|-\rangle\langle+| + |+\rangle\langle-|-\rangle\langle+| + |-\rangle\langle+|+\rangle\langle-|\right]\right\rangle$$

$$= \left\langle\frac{\hbar^2}{4}\left[|+\rangle\langle+| + |-\rangle\langle-|\right]\right\rangle = \frac{\hbar^2}{4}\left[\langle+|+\rangle\langle+|+\rangle + \cancel{\langle+|-\rangle\langle-|+\rangle}\right] = \frac{\hbar^2}{4}$$

- $\langle S_y\rangle^2 = \langle+|S_y|+\rangle^2 = \left[i\langle+|+\rangle\cancel{\langle-|+\rangle} + i\cancel{\langle+|-\rangle}\langle+|+\rangle\right]^2 = 0$

Thus: $\langle(\Delta S_y)^2\rangle = \frac{\hbar^2}{4} - 0 = \frac{\hbar^2}{4}$ ✓

We also need $[S_x, S_y] = S_xS_y - S_yS_x$

$$= \frac{\hbar^2}{4}(|+\rangle\langle-| + |-\rangle\langle+|)(-i|+\rangle\langle-| + i|-\rangle\langle+|) - \frac{\hbar^2}{4}(-i|+\rangle\langle-| + i|-\rangle\langle+|)(|+\rangle\langle-| + |-\rangle\langle+|)$$

$$= \frac{\hbar^2}{4}\Big[-i|+\rangle\langle-|+\rangle\langle-| - i|-\rangle\langle+|+\rangle\langle-| + i|+\rangle\langle-|-\rangle\langle+| + i|-\rangle\langle+|-\rangle\langle+|$$
$$+ i|+\rangle\langle-|+\rangle\langle-| - i|-\rangle\langle+|+\rangle\langle-| + i|+\rangle\langle-|-\rangle\langle+| - i|-\rangle\langle+|-\rangle\langle+|\Big]$$

$$= \frac{\hbar^2}{4}\left[-i|-\rangle\langle-| + i|+\rangle\langle+| - i|-\rangle\langle-| + i|+\rangle\langle+|\right]$$

$$= \frac{i\hbar^2}{2}\left[|+\rangle\langle+| - |-\rangle\langle-|\right]$$

$$\langle[S_x, S_y]\rangle = \frac{i\hbar^2}{2}\left[\langle+|+\rangle\langle+|+\rangle - \langle+|-\rangle\langle-|+\rangle\right] = \frac{i\hbar^2}{2}$$

$$|\langle[S_x, S_y]\rangle|^2 = \frac{i\hbar^2}{2}\left(\frac{-i\hbar^2}{2}\right) = \frac{\hbar^4}{4}$$ ✓

Meanwhile $\langle(\Delta S_x)^2\rangle\langle(\Delta S_y)^2\rangle = \frac{\hbar^2}{4}\cdot\frac{\hbar^2}{4} = \frac{\hbar^4}{16}$

$\frac{1}{4}|\langle[S_x, S_y]\rangle|^2 = \frac{\hbar^4}{16}$

Thus $\langle(\Delta S_x)^2\rangle\langle(\Delta S_y)^2\rangle \geq \frac{1}{4}|\langle[S_x, S_y]\rangle|^2$ holds

because $\hbar^4/16 \geq \hbar^4/16$ ✓

(b) Now we must repeat the check in pt. (a) w/ respect to the $|S_x+\rangle$ state.

$\rightarrow \; S_x^2 = \frac{\hbar^2}{4}\left[|+\rangle\langle+| + |-\rangle\langle-| \right]$ from pt. a.

$\langle S_x^2 \rangle = \frac{\hbar^2}{4}\left[\langle S_x+|+\rangle\langle+|S_x+\rangle + \langle S_x+|-\rangle\langle-|S_x+\rangle \right]$

note: $|S_x+\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle + |-\rangle \right)$

$= \frac{\hbar^2}{4}\left[\frac{1}{2}\left(\langle+|+\rangle + \langle-|+\rangle \right)\left(\langle+|+\rangle + \langle+|-\rangle \right) + \frac{1}{2}\left(\langle+|-\rangle + \langle-|-\rangle \right)\left(\langle-|+\rangle + \langle-|-\rangle \right) \right]$

$= \frac{\hbar^2}{8}\left[1 + 1 \right] = \frac{\hbar^2}{4}$

$\langle S_x \rangle^2 = \left[\frac{\hbar}{2}\left(\langle S_x+|+\rangle\langle-|S_x+\rangle + \langle S_x+|-\rangle\langle+|S_x+\rangle \right) \right]^2$

$= \frac{\hbar^2}{4}\left[\frac{1}{2}\left(\langle+|+\rangle + \langle-|+\rangle \right)\left(\langle-|+\rangle + \langle-|-\rangle \right) + \frac{1}{2}\left(\langle+|-\rangle + \langle-|-\rangle \right)\left(\langle+|+\rangle + \langle+|-\rangle \right) \right]^2$

$= \frac{\hbar^2}{4}\left[1 \right]^2 = \frac{\hbar^2}{4}$

$\therefore \; \langle (\Delta S_x)^2 \rangle = \frac{\hbar^2}{4} - \frac{\hbar^2}{4} = 0$ ✓

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$\rightarrow \; |\langle [S_x, S_y] \rangle|^2 = |\langle \frac{i\hbar^2}{2}\left[|+\rangle\langle+| - |-\rangle\langle-| \right] \rangle|^2$ from pt. (a).

$= \frac{\hbar^4}{4}\left| \langle S_x+|+\rangle\langle+|S_x+\rangle - \langle S_x+|-\rangle\langle-|S_x+\rangle \right|^2$

$= \frac{\hbar^4}{4}\left| \frac{1}{2}\left(\langle+|+\rangle + \langle-|+\rangle \right)\left(\langle+|+\rangle + \langle+|-\rangle \right) - \frac{1}{2}\left(\langle+|-\rangle + \langle-|-\rangle \right)\left(\langle-|+\rangle + \langle-|-\rangle \right) \right|^2$

$= \frac{\hbar^4}{4}\left| \frac{1}{2} - \frac{1}{2} \right|^2 = 0$ ✓

$\therefore \; \overset{0}{\langle (\Delta S_x)^2 \rangle}\langle (\Delta S_y)^2 \rangle \geq \overset{0}{\frac{1}{4}|\langle [S_x, S_y] \rangle|^2}$

b/c $0 \geq 0$. Note that we don't know/don't care about $\langle (\Delta S_y)^2 \rangle$. $\langle (\Delta S_x)^2 \rangle\langle (\Delta S_y)^2 \rangle = 0$. ✓