Sakurai 1-17

Given: $[A_1, A_2] \neq 0$, and A_1, A_2 are observables.
H is the Hamiltonian, and $[A_1, H] = 0$, $[A_2, H] = 0$

We are asked to prove that the energy eigenstates of A_1 and A_2 exhibit some degeny. In other words prove that $[A_1, A_2] \neq 0$ implies there is some degeny.

Proof by contrapositive. Assume there is no degeny. Prove $[A_1, A_2] = 0$
we know $HA_1 = A_1 H$ and $HA_2 = A_2 H$.

Thus $HA_1|E\rangle = E(A_1|E\rangle)$ ✓
$HA_2|E\rangle = E(A_2|E\rangle)$ ✓ → where $|E\rangle$ is an energy eigenstate of H w/ eigenvalue E.

Thus, since H and A_1 (and H and A_2) commute, they share a complete set of eigenstates. In other words $|E\rangle$ is an eigenstate of both A_1 and A_2. We should really write $|E, a_1, a_2\rangle$ so

$A_1|E, a_1, \cancel{a_2}\rangle = a_1|E, a_1, \cancel{a_2}\rangle$

$A_2|E, \cancel{a_1}, a_2\rangle = a_2|E, \cancel{a_1}, a_2\rangle$ → Usually you only put "good" quantum numbers in the kets Since A_1 and A_2 do not commute, you cannot have simultaneous eigenstates and you only write either a_1 or a_2 in the ket.)

But for simplicity, I'll just use $|E\rangle$. ← (unless you're assuming no assuming no degeny degeneracy)

Now notice:

$$0|E\rangle = (a_1 a_2 - a_2 a_1)|E\rangle = (A_1 A_2 - A_2 A_1)|E\rangle = [A_1, A_2]|E\rangle$$

Thus $[A_1, A_2] = 0$ ✓. Thus degeneracy must exist. □

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An "exception" (or a case in which there is no degeny) is L^2 operating on $|l=0, m=0\rangle$

$$L^2|l=0, m=0\rangle = \hbar^2 l(l+1)|l=0, m=0\rangle = 0|l=0, m=0\rangle.$$

✓ The eigenvalue of L^2 w/ respect to eigenstate $|l=0, m=0\rangle$ is 0.
There are NO other states that, when acted upon by operator L^2 return 0.

$\left(\begin{array}{l} l > 0 \Rightarrow l(l+1) > 0 \text{ as well.} \\ m \neq 0 \Rightarrow l \neq 0. \end{array} \right)$

→ More generally, if $A_1|E\rangle = 0$ or $A_2|E\rangle = 0$, the proof falls apart.