7.7 ① find $\left.\frac{\partial^2 P}{\partial T^2}\right|_v$. First note that eq. 7.1.15 gives us $C_v = \frac{3}{2} V \frac{\partial P}{\partial T}$

Thus $\left.\frac{\partial^2 P}{\partial T^2}\right|_v = \frac{2}{3V} \left.\frac{\partial C_v}{\partial T}\right|_v$

And luckily $\left.\frac{\partial C_v}{\partial T}\right|_v$ is given in the problem 7.6 statment.

Shorthand: $g_v \equiv g_v(z)$ to save space

$$\boxed{\begin{aligned} \left.\frac{\partial^2 P}{\partial T^2}\right|_v &= \frac{2}{3}\frac{Nk}{TV}\left[\frac{45}{8}\frac{g_{5/2}}{g_{3/2}} - \frac{9}{4}\frac{g_{3/2}}{g_{1/2}} - \frac{27}{8}\frac{g_{3/2}^2\, g_{-1/2}}{g_{1/2}^3}\right] \quad & T > T_c \\ \left.\frac{\partial^2 P}{\partial T^2}\right|_v &= \frac{2}{3}\frac{Nk}{TV}\left[\frac{45}{8}\frac{v}{\lambda^3}\zeta\left(\tfrac{5}{2}\right)\right] \quad & T < T_c \end{aligned}}$$

② find $\left.\frac{\partial^2 \mu}{\partial T^2}\right|_v$ First note that $\mu = kT\ln(z)$, so $\left.\frac{\partial \mu}{\partial T}\right|_v = k\ln(z) + \frac{kT}{z}\left.\frac{\partial z}{\partial T}\right|_v$

By 7.1.36, $\left.\frac{\partial \mu}{\partial T}\right|_v = k\ln(z) + kT\left(-\frac{3}{2T}\frac{g_{3/2}}{g_{1/2}}\right) = k\ln(z) - \frac{3k}{2}\frac{g_{3/2}}{g_{1/2}}$ for $T > T_c$

Then the second derivative...

$$\left.\frac{\partial^2 \mu}{\partial T^2}\right|_v = \frac{\partial}{\partial T}\left[k\ln(z) - \frac{3}{2}k\frac{g_{3/2}}{g_{1/2}}\right] = \frac{\partial}{\partial T}k\ln(z) - \frac{3}{2}k\frac{\partial}{\partial T}\left(\frac{g_{3/2}}{g_{1/2}}\right)$$

For the second term, use the quotient rule

$$\frac{\partial}{\partial T}\left(\frac{g_{3/2}}{g_{1/2}}\right) = \frac{\frac{\partial}{\partial T}(g_{3/2})\,g_{1/2} - g_{3/2}\frac{\partial}{\partial T}(g_{1/2})}{(g_{1/2})^2}$$

Note that $\left.\frac{\partial}{\partial T}\right|_v = \frac{\partial}{\partial(\ln z)}\left.\frac{\partial(\ln z)}{\partial T}\right|_v$ and by D.10, $\frac{\partial}{\partial(\ln z)} g_v(z) = g_{v-1}(z)$

$$\Rightarrow \frac{\partial}{\partial T}\left(\frac{g_{3/2}}{g_{1/2}}\right) = \left.\frac{\partial(\ln z)}{\partial T}\right|_v\left(\frac{g_{1/2}^2 - g_{3/2}\,g_{-1/2}}{(g_{1/2})^2}\right)$$

Put it all back in:

$$\left.\frac{\partial^2 \mu}{\partial T^2}\right|_v = \frac{\partial \ln z}{\partial T}\left[k - \frac{3}{2}k\left(\frac{g_{1/2}^2 - g_{3/2}\,g_{-1/2}}{(g_{1/2})^2}\right)\right] = k\frac{\partial \ln z}{\partial T}\left[1 - \frac{3}{2} + \frac{3}{2}\frac{g_{3/2}\,g_{-1/2}}{(g_{1/2})^2}\right]$$

$$= \frac{k\,\partial \ln z}{\partial T}\left[\frac{3}{2}\frac{g_{3/2}\,g_{-1/2}}{g_{1/2}^2} - \frac{1}{2}\right]$$

Last, $\frac{\partial \ln z}{\partial T} = \frac{\partial}{\partial z}\cdot\frac{\partial z}{\partial T}\cdot\ln(z) = \frac{1}{z}\frac{\partial z}{\partial T} = -\frac{3}{2}\frac{g_{3/2}}{g_{1/2}}$ by 7.1.36. so,

$$\left.\frac{\partial^2 \mu}{\partial T^2}\right|_v = \frac{-3k}{2T}\frac{g_{3/2}}{g_{1/2}}\left[\frac{3}{2}\frac{g_{3/2}\,g_{-1/2}}{g_{1/2}^2} - \frac{1}{2}\right] \Rightarrow \boxed{\left.\frac{\partial^2 \mu}{\partial T^2}\right|_v = -\frac{9}{4}\frac{k}{T}\frac{(g_{3/2}(z))^2\,g_{-1/2}(z)}{(g_{1/2}(z))^2} + \frac{3}{4}\frac{k}{T}\frac{g_{3/2}(z)}{g_{1/2}(z)}}$$

for $T > T_c$

7.7, cont

(3) find $\left.\frac{\partial^2 \mu}{\partial T^2}\right|_P$ once again, $\left.\frac{\partial \mu}{\partial T}\right|_P = k\ln(z) + \frac{kT}{z}\left.\frac{\partial z}{\partial T}\right|_P$

So, let's get $\left.\frac{\partial z}{\partial T}\right|_P$ By 7.1.7, $P = \frac{kT}{\lambda^3} g_{5/2}(z) = \frac{kT^{5/2}(2\pi m k)^{3/2}}{h^3} \frac{\partial}{\partial T} g_{5/2}(z)$

$$\left.\frac{dP}{dT}\right|_P = 0 = \frac{5}{2}\frac{kT^{5/2}(2\pi m k)^{3/2}}{h^3} g_{5/2}(z) + \frac{\cancel{k}T^{5/2}(2\pi m k)^{3/2}}{h^3}\left.\frac{\partial}{\partial T} g_{5/2}(z)\right|_P$$

$$0 = \frac{k^{5/2}(2\pi m)^{3/2}}{h^3}\left[\frac{5}{2}T^{3/2} g_{5/2}(z) + T^{5/2}\left.\frac{\partial z}{\partial T}\right|_P \frac{\partial}{\partial z} g_{5/2}(z)\right.$$

$$\Rightarrow \left.\frac{\partial z}{\partial T}\right|_P = \frac{-\frac{5}{2}T^{3/2} g_{5/2}(z)}{T^{5/2}\frac{\partial g_{5/2}(z)}{\partial z}} = \frac{-5 g_{5/2}(z)}{2T \frac{\partial}{\partial z} g_{5/2}(z)} = \frac{-z5 g_{5/2}(z)}{2T g_{3/2}(z)} \quad ☆$$

$$\Rightarrow \left.\frac{\partial \mu}{\partial T}\right|_P = k\ln(z) - \frac{5}{2}k\frac{g_{5/2}(z)}{g_{3/2}(z)}$$

Now we'll take the second derivative.

$$\left.\frac{\partial^2 \mu}{\partial T^2}\right|_P = k\left.\frac{\partial \ln z}{\partial T}\right|_P\left(1 - \frac{5}{2}\frac{\partial}{\partial \ln z}\left(\frac{g_{5/2}}{g_{3/2}}\right)\right) = k\left.\frac{\partial \ln z}{\partial T}\right|_P\left(1 - \frac{5}{2} + \frac{5}{2}\frac{g_{5/2}\, g_{1/2}}{g_{3/2}^2}\right)$$

$$= k\left.\frac{\partial \ln z}{\partial T}\right|_P\left(\frac{5}{2}\frac{g_{5/2}\, g_{1/2}}{g_{3/2}^2} - \frac{3}{2}\right) = \frac{k}{z}\left.\frac{\partial z}{\partial T}\right|_P\left(\frac{5}{2}\frac{g_{5/2}\, g_{1/2}}{g_{3/2}^2} - \frac{3}{2}\right)$$

Now using ☆ to plug into $\left.\frac{\partial z}{\partial T}\right|_P$, we get

$$= k\frac{5}{2T}\frac{g_{5/2}}{g_{3/2}}\left(\frac{3}{2} - \frac{5}{2}\frac{g_{5/2}\, g_{1/2}}{g_{3/2}^2}\right) \Rightarrow \boxed{\left.\frac{\partial^2 \mu}{\partial T^2}\right|_P = \frac{15}{4}\frac{k}{T}\frac{g_{5/2}(z)}{g_{3/2}(z)} - \frac{25}{4}\frac{k}{T}\frac{g_{5/2}^2(z)\, g_{1/2}(z)}{g_{3/2}^3(z)}}$$

(4) check that $C_V = VT\left.\frac{\partial^2 P}{\partial T^2}\right|_V - NT\left.\frac{\partial^2 \mu}{\partial T^2}\right|_V$

Plug in the second derivatives from (1) and (2):

$$C_V = \frac{2}{3}Nk\left[\frac{45}{8}\frac{g_{5/2}}{g_{3/2}} - \frac{9}{4}\frac{g_{3/2}}{g_{1/2}} - \frac{27}{8}\frac{g_{3/2}^2\, g_{-1/2}}{g_{1/2}^3}\right] + \frac{9}{4}Nk\frac{g_{3/2}\, g_{-1/2}}{g_{1/2}^2} - \frac{3}{4}Nk\frac{g_{3/2}}{g_{1/2}}$$

$$= \frac{15}{4}Nk\frac{g_{5/2}}{g_{3/2}} - \frac{3}{2}Nk\frac{g_{3/2}}{g_{1/2}} - \cancel{\frac{9}{4}Nk\frac{g_{3/2}^2\, g_{-1/2}}{g_{1/2}}} + \cancel{\frac{9}{4}Nk\frac{g_{3/2}^2\, g_{-1/2}}{g_{1/2}}} - \frac{3}{4}Nk\frac{g_{3/2}}{g_{1/2}}$$

$$\boxed{C_V = \frac{15}{4}Nk\frac{g_{5/2}(z)}{g_{3/2}(z)} - \frac{9}{4}Nk\frac{g_{3/2}}{g_{1/2}}}$$

✓ This agrees with 7.1.37.

(5) check that $C_P = -NT\left.\frac{\partial^2 \mu}{\partial T^2}\right|_P$

First, ~~fom~~ 7.1.48b, $C_P = C_V \cdot \frac{5}{3}\frac{g_{5/2}\, g_{1/2}}{g_{3/2}^2} = \frac{25}{4}Nk\frac{g_{5/2}^2\, g_{1/2}}{g_{3/2}^3} - \frac{15}{4}Nk\frac{g_{5/2}}{g_{3/2}}$

where I plugged in the result of (4) for C_V

Now check using the result of (3)

$$-NT\left.\frac{\partial^2 \mu}{\partial T^2}\right|_P = -Nk\frac{15}{4}\frac{g_{5/2}}{g_{3/2}} + Nk\frac{25}{4}\frac{g_{5/2}^2\, g_{1/2}}{g_{3/2}^3} = \boxed{\frac{25}{4}Nk\frac{g_{5/2}^2(z)\, g_{1/2}(z)}{g_{3/2}^3(z)} - \frac{15}{4}Nk\frac{g_{5/2}(z)}{g_{3/2}(z)} = C_P} ✓$$

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How do these behave going to T_c?

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