$\boxed{7.20}$

$A = -kT \ln Q$, and per the problem statement, $\ln Q \approx \int_0^\infty \ln(Q_1(\omega,T))\, g(\omega)\, d\omega$

Plugging in 3.8.14, and 7.3.2, we get

$$A = -kT \int_0^\infty \ln\left[\frac{e^{-1/2\,\beta\hbar\omega}}{1-e^{-\beta\hbar\omega}}\right] \frac{V\omega^2}{\pi^2 c^3}\, d\omega$$

Note that 7.3.2 gives $\frac{g(\omega)}{V} = \frac{\omega^2}{\pi^2 c^3}$ (Per unit volume.) $\Rightarrow g(\omega)\,d\omega = \frac{\omega^2 V\, d\omega}{\pi^2 c^3}$

$$= \frac{-kTV}{\pi^2 c^3}\left[\frac{1}{2}\beta\hbar \int_0^\infty \omega^3 d\omega + \int_0^\infty d\omega\, \omega^2 \ln(1-e^{-\beta\hbar\omega})\right]$$

Let's Do term #2 first

u-substitution: $x = \beta\hbar\omega$, $dx = \beta\hbar\, d\omega$, $d\omega = \frac{dx}{\beta\hbar} \Rightarrow \omega^2 = \frac{x^2}{\beta^2\hbar^2}$

$$\int_0^\infty d\omega\, \omega^2 \ln(1-e^{-\beta\hbar\omega}) \rightarrow \frac{1}{(\beta\hbar)^3}\int_0^\infty \underbrace{dx\, x^2}_{dv}\, \underbrace{\ln(1-e^{-x})}_{u}$$

Integrate by parts

$dv = dx\, x^2 \Rightarrow \frac{dv}{dx} = x^2 \Rightarrow v = \frac{1}{3}x^3$

$u = \ln(1-e^{-x}) \Rightarrow \frac{du}{dx} = \frac{e^{-x}}{1-e^{-x}} \Rightarrow du = \frac{e^{-x}}{1-e^{-x}}\, dx$

$$= \frac{1}{(\beta\hbar)^3}\left[\ln(1-e^{-x})\frac{1}{3}x^3\Big|_0^\infty - \int_0^\infty \frac{1}{3}x^3 \frac{e^{-x}}{1-e^{-x}}\, dx\right]$$

0 (b/c $\ln(1) = 0$ and $0^3 = 0$)

$$= \frac{-1}{(\beta\hbar)^3}\int_0^\infty \frac{1}{3}x^3 \frac{e^{-x}}{1-e^{-x}}\, dx = \frac{-1}{3(\beta\hbar)^3}\int_0^\infty \frac{x^3}{e^x - 1}\, dx = \frac{-\Gamma(4)}{3(\beta\hbar)^3}\frac{1}{\Gamma(4)}\int_0^\infty \frac{x^{4-1}}{(1)^{-1}e^x - 1}\, dx$$

Now regonize that $\frac{1}{\Gamma(4)}\int_0^\infty \frac{x^{4-1}}{(1)^{-1}e^x-1}\, dx = g_4(1) = \zeta(4)$ (the reimann zeta fn.)

$$= \frac{-\Gamma(4)\zeta(4)}{3(\beta\hbar)^3} = \frac{-3!}{3(\beta\hbar)^3}\frac{\pi^4}{90} = \frac{-\pi^4}{45(\beta\hbar)^3}$$

appendix D.

Now plug this term back into A (and carry out the integration symbolically in term #1):

$$A = \frac{kTV}{\pi^2 c^3}\left[\frac{1}{2}\beta\hbar\frac{\omega^4}{4}\Big|_0^\infty - \frac{\pi^4}{45}\frac{1}{(\beta\hbar)^3}\right] = \lim_{\omega\to\infty}\left[\frac{V}{\beta\pi^2 c^3}\cdot\frac{1}{8}\frac{\beta\hbar\omega^4}{4}\right] - \frac{V}{\beta\pi^2 c^3}\frac{\pi^4}{45}\frac{1}{(\beta\hbar)^3}$$

$$A = \lim_{\omega\to\infty}\left[\frac{V\hbar\omega^4}{32\pi^2 c^3}\right] - \frac{V\pi^2}{45(c\hbar)^3\beta^4}$$

The first term is a constant, and we can always shift energies by a constant. Thus...

$$\boxed{A = \frac{-V\pi^2}{45(c\hbar)^3\beta^4}}$$

Thermodynamics

$$P = -\left(\frac{\partial A}{\partial V}\right)_{T,N} = \boxed{\frac{\pi^2}{45(c\hbar)^3\beta^4} = P}$$

$$U = -T^2\left[\frac{\partial}{\partial T}\left(\frac{A}{T}\right)\right]_{N,V} = -T^2\frac{\partial}{\partial T}\left(\frac{-V\pi^2 k^4 T^3}{45(c\hbar)^3}\right) = T^2\left(\frac{3V\pi^2 k^4 T^2}{45(c\hbar)^3}\right) = \frac{V\pi^2}{15c^3\hbar^3\beta^4}$$

$$\Rightarrow \boxed{\frac{U}{V} = \frac{\pi^2}{15(c\hbar)^3\beta^4}}$$ This agrees with 7.3.12

$$\boxed{\mu = \left(\frac{\partial A}{\partial N}\right)_{T,V} = 0} \qquad S = -\left(\frac{\partial A}{\partial T}\right)_{V,N} = \frac{\partial}{\partial T}\left[\frac{V\pi^2 k^4 T^4}{45(c\hbar)^3}\right] \Rightarrow \boxed{S = \frac{4V\pi^2 k^4 T^3}{45(c\hbar)^3}}$$