$\boxed{6.3}$ We are to find $\langle n_\varepsilon \rangle$ in the case where n_ε can be either 1 or 2 or.... or ℓ.

$\langle n_\varepsilon \rangle = -\frac{1}{\beta}\left(\frac{\partial q}{\partial \varepsilon}\right)_\varepsilon$ by 6.2.22, and $q(z,V,T) = \ln \mathcal{Q}(z,V,T)$ by 6.2.17.

Thus, the main task is to find $\mathcal{Q}(z,V,T)$.

By 6.2.15 $\mathcal{Q}(z,V,T) = \left[\sum_{n_0=1}^{\ell} (ze^{-\beta\varepsilon_0})^{n_0}\right]\left[\sum_{n_1=1}^{\ell} (ze^{-\beta\varepsilon_1})^{n_1}\right]\left[\cdots\right]\cdots\cdots$

This is a product of Geometric Series.

The sum of the first n terms of a geo. series $a + ar + ar^2 + ar^3 + \cdots + ar^{n-1}$ is $a\frac{1-r^n}{1-r}$

In this case $n \to \ell+1$, $a \to 1$, and $r \to ze^{-\beta\varepsilon}$

$$\mathcal{Q}(z,V,T) = \left[\frac{1-(ze^{-\beta\varepsilon_0})^{\ell+1}}{1-ze^{-\beta\varepsilon_0}}\right]\left[\frac{1-(ze^{-\beta\varepsilon_1})^{\ell+1}}{1-ze^{-\beta\varepsilon_1}}\right]\cdots\cdots = \prod_\varepsilon \frac{1-(ze^{-\beta\varepsilon})^{\ell+1}}{1-ze^{-\beta\varepsilon}}$$

$$q(z,V,T) = \ln \mathcal{Q}(z,V,T) = \sum_\varepsilon \ln\left(1-(ze^{-\beta\varepsilon})^{\ell+1}\right) - \sum_\varepsilon \ln\left(1-ze^{-\beta\varepsilon}\right)$$

$\langle n_\varepsilon \rangle = -\frac{1}{\beta}\left(\frac{\partial q}{\partial \varepsilon}\right)_\varepsilon$ (holding ε constant = take away the summations)

$$= -\frac{1}{\beta}\left[\frac{(\ell+1)\beta z(ze^{-\beta\varepsilon})^{\ell}}{e^{\beta\varepsilon} - z(ze^{-\beta\varepsilon})^{\ell}} - \frac{\beta z e^{-\beta\varepsilon}}{1-ze^{-\beta\varepsilon}}\right]$$

$$= \frac{ze^{-\beta\varepsilon}}{1-ze^{-\beta\varepsilon}} - \frac{\ell+1}{\frac{z^{-1}e^{\beta\varepsilon}}{(ze^{-\beta\varepsilon})^{\ell}} - \cancel{\frac{z(ze^{-\beta\varepsilon})^{\ell}}{z(ze^{-\beta\varepsilon})^{\ell}}}1}$$

$$= \frac{1}{z^{-1}e^{\beta\varepsilon}-1} - \frac{\ell+1}{z^{-(\ell+1)}e^{-\beta\varepsilon(\ell+1)}-1} \rightarrow \boxed{\langle n_\varepsilon \rangle = \frac{1}{z^{-1}e^{\beta\varepsilon}-1} - \frac{\ell+1}{(z^{-1}e^{\beta\varepsilon})^{\ell+1}-1}}$$

By 6.3.22, $\langle n_\varepsilon \rangle = \frac{1}{z^{-1}e^{\beta\varepsilon}+a}$ where $a = \begin{cases} 1 & \text{in the fermi-dirac case} \\ -1 & \text{in Bose-Einstein case.} \end{cases}$

We check this now using the above boxed expression.

$$\ell=1:\quad \langle n_\varepsilon \rangle = \frac{1}{z^{-1}e^{\beta\varepsilon}-1} - \frac{2}{(z^{-1}e^{\beta\varepsilon})^2-1} = \frac{1}{z^{-1}e^{\beta\varepsilon}-1} - \frac{2}{(z^{-1}e^{\beta\varepsilon}-1)(z^{-1}e^{\beta\varepsilon}+1)}$$

$$= \frac{(z^{-1}e^{\beta\varepsilon}+1)-2}{(z^{-1}e^{\beta\varepsilon}-1)(z^{-1}e^{\beta\varepsilon}+1)} = \frac{\cancel{z^{-1}e^{\beta\varepsilon}-1}}{\cancel{(z^{-1}e^{\beta\varepsilon}-1)}(z^{-1}e^{\beta\varepsilon}+1)} = \boxed{\frac{1}{z^{-1}e^{\beta\varepsilon}+1} = \langle n_\varepsilon \rangle_{F.D.}}$$

$$\ell \to \infty:\quad \langle n_\varepsilon \rangle = \frac{1}{z^{-1}e^{\beta\varepsilon}-1} - \cancel{\frac{\ell+1}{(z^{-1}e^{\beta\varepsilon})^{\ell+1}-1}} \to 0$$

The second term goes to zero b/c e^ℓ grows faster than ℓ as $\ell \to \infty$.

$$\boxed{\langle n_\varepsilon \rangle_{B.E.} = \frac{1}{z^{-1}e^{\beta\varepsilon}-1}}$$

These check out ✓