$\boxed{6.19}$ Probability that 2 molecules chosen @ random from a Maxwellian Gas will have total energy between E, and E+dE.

$$P(E)\,dE \propto e^{-\beta E} g(E)\,dE \quad \text{by } 3.4.3. \quad \text{(unormalized)}.$$

we have two molecules, so the probability distribution is the product of the two one-particle distributions.

$$\rho(\varepsilon_2)\rho(\varepsilon_1)\,d\varepsilon_1\,d\varepsilon_2 \propto g_1(\varepsilon_1)\,g_2(\varepsilon_2)\,e^{-\beta\varepsilon_1} e^{-\beta\varepsilon_2}\,d\varepsilon_1\,d\varepsilon_2$$

[In general, when the Energy spectrum is of the form $\varepsilon \sim p^s$, then in d dimensions, $d^dp \sim d\varepsilon\, p^{d-s} = d\varepsilon\, \varepsilon^{\frac{d-s}{s}}$]

Thus, since for a maxwellian gas $E \sim p^2$ (i.e. $\frac{p^2}{2m}$),

$$g_1(\varepsilon_1) = \varepsilon_1^{1/2}, \quad g_2(\varepsilon_2) = \varepsilon_2^{1/2}$$

$$\Rightarrow \rho(\varepsilon_1)\rho(\varepsilon_2)\,d\varepsilon_1 d\varepsilon_2 \propto e^{-\beta\varepsilon_1} e^{-\beta\varepsilon_2}\,\varepsilon_1^{1/2}\varepsilon_2^{1/2}\,d\varepsilon_1\,d\varepsilon_2$$

At this point we want to rewrite one of the ε's in terms of E, so we can integrate out the other, and be left with a probability density dependent only on E.

u-sub $\varepsilon_2 = E - \varepsilon_1 \Rightarrow \frac{d\varepsilon_2}{dE} = 1 \Rightarrow d\varepsilon_2 = dE$

$$\Rightarrow \rho(\varepsilon_1)\rho(\varepsilon_2)\,d\varepsilon_1 d\varepsilon_2 \propto \cancel{e^{-\beta\varepsilon_1}}\, e^{-\beta E}\, \cancel{e^{\beta\varepsilon_1}}\, \varepsilon_1^{1/2}(E-\varepsilon_1)^{1/2}\,d\varepsilon_1\,dE$$

$$P(E)\,dE \propto \int_0^E e^{-\beta E}\left[\varepsilon_1(E-\varepsilon_1)\right]^{1/2} d\varepsilon_1\,dE$$

where I'm only integrating over ε_1. Note that ε_1 ranges from 0 to E.

$$\boxed{P(E)\,dE \propto \frac{1}{8}\pi e^{-\beta E} E^2\,dE}$$

More Precisely, we use 3.5.16 with N=2: $g(E) = \frac{1}{2}\frac{V^2}{h^6}\frac{(2\pi m)^3}{2}E^2$

$$\Rightarrow \boxed{P(E)\,dE = 2\left(\frac{V^{2/3}\pi m}{h^2}\right)^3 e^{-\beta E} E^2\,dE}$$

Verify $\langle E\rangle = 3kT$

$$\langle E\rangle = \frac{\int_0^\infty E\,P(E)\,dE}{\int_0^\infty P(E)\,dE} = \frac{\int_0^\infty E^3 e^{-\beta E}\,dE}{\int_0^\infty E^2 e^{-\beta E}\,dE} \overset{\text{Gaussian Integral}}{=} \frac{\frac{\Gamma(4)}{\beta^4}}{\frac{\Gamma(3)}{\beta^3}} = \frac{6\beta^3}{2\beta^4} = \boxed{3kT}$$