$\boxed{5.1}$ The pauli spin matricies in the normal representation are

$$\hat{\sigma}_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \hat{\sigma}_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \qquad \hat{\sigma}_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

we can convert them into the new representation using the unitary operator

$$\hat{U} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \implies \hat{\sigma}_x' = \hat{U}\hat{\sigma}_x\hat{U}^{+} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \text{ similarly, } \hat{\sigma}_y' = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \hat{\sigma}_z' = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

The hamiltonian in the new representation is

$\hat{H}' = -\mu_B(\vec{\sigma}' \cdot \vec{B})$, and we assume $\vec{B} = B\hat{z}$, then $\hat{H}' = -\mu_B B \hat{\sigma}_z'$

$$\implies \hat{H}' = -\mu_B B \hat{U}\hat{\sigma}_z\hat{U}^{+}$$

$$\implies \hat{\rho}' = \frac{e^{\beta \hat{H}'}}{tr(e^{\beta \hat{H}'})} = \frac{e^{\beta\mu_B B \hat{U}\hat{\sigma}_z\hat{U}^{+}}}{tr(e^{\beta\mu_B B \hat{U}\hat{\sigma}_z\hat{U}^{+}})}$$

Lemma: $e^{\hat{U}\hat{\sigma}_z\hat{U}^{+}} = \hat{U}e^{\hat{\sigma}_z}\hat{U}^{+}$

Proof:
$$e^{\hat{U}\hat{\sigma}_z\hat{U}^{+}} = 1 + \hat{U}\hat{\sigma}_z\hat{U}^{+} + \frac{1}{2!}(\hat{U}\hat{\sigma}_z\hat{U}^{+})^2 + \frac{1}{3!}(\hat{U}\hat{\sigma}_z\hat{U}^{+})^3 + \cdots$$
$$= 1 + \hat{U}\hat{\sigma}_z\hat{U}^{+} + \frac{1}{2!}(\hat{U}\hat{\sigma}_z\cancel{\hat{U}^{+}\hat{U}}\hat{\sigma}_z\hat{U}^{+}) + \frac{1}{3!}(\hat{U}\hat{\sigma}_z\cancel{\hat{U}^{+}\hat{U}}\hat{\sigma}\cancel{\hat{U}^{+}\hat{U}}\hat{\sigma}\hat{U}^{+}) + \cdots$$
$$= U(1 + \hat{\sigma}_z + \frac{1}{2!}\hat{\sigma}_z^2 + \frac{1}{3!}\hat{\sigma}_z^3 + \cdots)U^{+} = \hat{U}e^{\hat{\sigma}_z}\hat{U}^{+} \quad \square$$

$$\therefore \hat{\rho}' = \frac{\hat{U}e^{\beta\mu_B B\hat{\sigma}_z}\hat{U}^{+}}{tr(\hat{U}e^{\beta\mu_B B\hat{\sigma}_z}\hat{U}^{+})}$$

also,
$$e^{\beta\mu_B B\hat{\sigma}_z} = \exp\begin{bmatrix} \beta\mu_B B & 0 \\ 0 & -\beta\mu_B B \end{bmatrix} = \begin{bmatrix} e^{\beta\mu_B B} & 0 \\ 0 & e^{-\beta\mu_B B} \end{bmatrix}$$

$$\implies \hat{U}e^{\beta\mu_B B\hat{\sigma}_z}\hat{U}^{+} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} e^{\beta\mu_B B} & 0 \\ 0 & e^{-\beta\mu_B B} \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} \cosh(\beta\mu_B B) & -\sinh(\beta\mu_B B) \\ -\sinh(\beta\mu_B B) & \cosh(\beta\mu_B B) \end{bmatrix}$$

and $tr(\hat{U}e^{\beta\mu_B B\hat{\sigma}_z}\hat{U}^{+}) = \cosh(\beta\mu_B B)$

$$\implies \hat{\rho}' = \frac{1}{\cosh(\beta\mu_B B)}\frac{1}{2}\begin{bmatrix} \cosh(\beta\mu_B B) & -\sinh(\beta\mu_B B) \\ -\sinh(\beta\mu_B B) & \cosh(\beta\mu_B B) \end{bmatrix} \implies \boxed{\hat{\rho}' = \frac{1}{2}\begin{bmatrix} 1 & -\tanh(\beta\mu_B B) \\ \tanh(\beta\mu_B B) & 1 \end{bmatrix}}$$

$$\langle \sigma_z' \rangle = tr(\hat{\rho}, \hat{\sigma}_z'\rangle = tr\left(\begin{bmatrix} \frac{1}{2} & -\tanh(\beta\mu_B B) \\ \tanh(\beta\mu_B B) & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\right) = tr\begin{bmatrix} \frac{1}{2}\tanh(\beta\mu_B B) & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2}\tanh(\beta\mu_B B) \end{bmatrix}$$

$\boxed{\langle \sigma_z' \rangle = \tanh(\beta\mu_B B)}$ ... same as 5.3.4

10/10