Pathria 2-7 (i) The first goal is to determine an "asymptotic" expression for Ω N 1-D SHO's, with energies $\varepsilon(n_r) = (n_r + \frac{1}{2})\hbar\omega_0$

Total energy of system $E = \sum_r \varepsilon(n_r) = \sum_r (n_r + \frac{1}{2})\hbar\omega_0 = \hbar\omega_0 \sum_r n_r + \frac{N}{2}\hbar\omega_0$

$\therefore$

$$\sum_r n_r = \frac{E}{\hbar\omega_0} + \frac{N}{2} \equiv R.$$ when $E \gg N$, $R \simeq \frac{E}{\hbar\omega_0}$

Ω is the number of ways to sum up the n_r's so that they equal R.

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2-7, cont. Following the procedure I used in question 1-8, we use 3.8.25:

$$\Omega = \frac{(R+N-1)!}{R!(N-1)!} \Rightarrow \ln\Omega = R\ln\left(\frac{R+N}{R}\right) + N\ln\left(\frac{R+N}{N}\right)$$

as before, where $R, N \gg 1$.

Further, in this book "assymptotically" meas $R \gg N$, so we have

$$\ln\Omega \approx \underbrace{R\ln\cancel{\left(\frac{R}{R}\right)}}_{0} + N\ln\left(\frac{R}{N}\right) = \ln\left(\left(\frac{R}{N}\right)^N\right) \approx \ln\left(\frac{R^N}{N!}\right)$$

Finally, $\Omega \approx \frac{R^N}{N!}$ Now plug in the circled expression for R on previous page $\Rightarrow$ $\boxed{\Omega \approx \frac{(E/\hbar\omega_0)^N}{N!}}$

(ii) We begin by writing the Hamiltonian of a single harmonic oscillator:

$$\varepsilon_r = \frac{p_r^2}{2m} + \frac{q_r^2}{2/m\omega_0^2}$$

Thus, for N oscillators we have $E = \sum_{r=1}^{N} \frac{p_r^2}{2m} + \frac{q_r^2}{2/m\omega_0^2}$

As you may have noticed, this ($\nearrow$) is just the equation of a 2N-Dimensional hyperelipse. We want to find the "Volume" of this phase space enclosed with total energy less than E. Though it would be very nice, I don't know how to integrate over this volume, so lets transform to a hypersphere.

let $p_r' = \frac{p_r}{\sqrt{2m}}$ and $q_r' = \frac{q_r}{\sqrt{2/m\omega_0^2}}$ $\left(\text{so } dp_r' = \left(\frac{1}{2m}\right)^{N/2} dp_r,\ dq_r' = \left(\frac{1}{2/m\omega_0^2}\right)^{N/2} dq_r\right)$

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Now, $E = \sum_{r=1}^{N}(p_r'^2 + q_r'^2) \rightarrow$ a hypersphe of radius $\sqrt{E}$

$$V(E) = \int_{0 \le \sum_{r=1}^{N} H \le E} \cdots \int d^N q\, d^N p \longrightarrow \int_{0 \le \sum_{r=1}^{N}(p_r'^2 + q_r'^2) \le E} \cdots\cdots \int d^N q'\, d^N p' \left(\frac{4m}{m\omega_0^2}\right)^{N/2}$$

$\left(\text{since } d^N q = \left(\frac{2}{m\omega_0^2}\right)^{N/2} d^N q' \text{ \& } d^N p = (2m)^{N/2} d^N p'\right)$

This allows us now to use eq. 7, appendix C:

$$V(E) = \left(\frac{4m}{m\omega_0^2}\right)^{N/2} \frac{\pi^{2N/2}}{(2N/2)!} \sqrt{E}^{2N} = \left(\frac{2}{\omega_0}\right)^N \frac{\pi^N E^N}{N!}$$

so $\boxed{V = \frac{(2\pi E/\omega_0)^N}{N!}}$

Now we note $V = C \cdot \Omega$ where C is a constant. That constant is clearly $\boxed{h^N}$