Pathria 1-8

We are given N quasiparticles w/ eigenvalues for their energies $\varepsilon(n) = hn\nu$ when $n = 0, 1, 2, \ldots$ and total energy E.
Our task is to find $\Omega(N,E)$, the multiplicity of microstates.

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Our N quasi-particles each have just one quantum number n_r, for a total of N degrees of freedom.
Thus

$$\sum_{r=1}^{N} n_r = E/h\nu \equiv R \quad \text{where} \quad E \equiv \sum_{r=1}^{N} \varepsilon(n_r)$$

Following the argument preceeding 3.8.25 on p. 69, the number of distinct ways of alloting R quanta (which are indistinguishable) into N distinguishable "boxes" is the same as the multiplicity: 3.8.25)

$$\Omega = \frac{(R+N-1)!}{R!\,(N-1)!} \Rightarrow \ln\Omega = \ln((R+N-1)!) - \ln R! - \ln((N-1)!)$$

Assympotically, we can use Stirlings Aprox ($\ln(a!) \simeq a\ln(a)$)

$$\ln\Omega \approx (R+N-1)\ln(R+N-1) - R\ln R - (N-1)\ln(N-1)$$

Furthermore, assymptotically, $R, N \gg 1$, so

$$\ln\Omega \simeq (R+N)\ln(R+N) - R\ln R - N\ln N = R\ln\left(\frac{R+N}{R}\right) + N\ln\left(\frac{R+N}{N}\right)$$

Plug in for R to get

$$\boxed{\ln\Omega \simeq \frac{E}{h\nu}\ln\left(\frac{E+Nh\nu}{E}\right) + N\ln\left(\frac{E+Nh\nu}{Nh\nu}\right)}$$

→ Now, we are to find $T(E/N, h\nu)$. We will use 1.2.4: $\left(\frac{\partial S}{\partial E}\right)_N = \frac{1}{T}$

$S = k\ln\Omega$, so $T = \dfrac{1}{k\frac{d}{dE}(\ln\Omega(E,N))}$

just use this formula, and $\ln\Omega$ above and use Mathematica (see attached) to get:

$$\boxed{T = \frac{h\nu}{k\ln\left(1 + \frac{h\nu}{E/N}\right)}}$$

→ Finally, what happens as $E/Nh\nu \gg 1$? well... $\frac{Nh\nu}{E} \ll 1$, so

$$\ln\left(1 + \frac{Nh\nu}{E}\right) \approx \frac{Nh\nu}{E}$$

$$\therefore T \approx \frac{h\nu}{k\frac{Nh\nu}{E}} = \boxed{\frac{E}{Nk}}$$

We end up with a simple expression for T in the high-E limit.